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Continuation sheaves in dynamics: Sheaf cohomology and bifurcation

K. Alex Dowling [a,*], William D. Kalies [b,1], Robert C.A.M. Vandervorst [c]

[a] *Rutgers University, 110 Frelinghusen Road, Piscataway, 08854, NJ, USA*
[b] *Florida Atlantic University, 777 Glades Road, Boca Raton, 33431, FL, USA*
[c] *VU University, De Boelelaan 1081a, Amsterdam, 1081, HV, the Netherlands*

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Abstract

Algebraic structures such as the lattices of attractors, repellers, and Morse representations provide a computable description of global dynamics. In this paper, a sheaf-theoretic approach to their continuation is developed. The algebraic structures are cast into a categorical framework to study their continuation systematically and simultaneously. Sheaves are built from this abstract formulation, which track the algebraic data as systems vary. Sheaf cohomology is computed for several classical bifurcations, demonstrating its ability to detect and classify bifurcations.

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* Corresponding author.
E-mail address: kenneth.dowling@rutgers.edu (K.A. Dowling).
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1. Introduction

Dynamical systems theory is the study of the structure of invariant sets, particularly those sets that govern the long-term behavior of a system. As a dynamical system is perturbed, the structure of these invariant sets can change qualitatively through bifurcation. The global asymptotic dynamics of a system can be described by the structure of its attractors. Indeed, in his fundamental decomposition theorem, Conley uses the set of all attractors in a system to establish a global decomposition into minimal (chain)-recurrent components and connecting orbits between them [1]. The algebraic structure that underlies this decomposition is codified in the fact that the set of all attractors naturally forms a bounded, distributive lattice [2,3]. In a series of papers [2–7] the theoretical framework for such dynamically meaningful algebraic structures has been developed. This framework has been used to design algorithms to compute global dynamical information rigorously for explicit systems [8–13] as well as models obtained from data [14–19].

These dynamical structures, such as the lattice of attractors, have two important characteristics.

1. They are algebraic invariants of the dynamics, and hence they are amenable to computations.
2. They are comprised of isolated invariant sets, and hence have isolating neighborhoods which are robust under perturbation.

Indeed, for a parametrized family of dynamical systems, invariant sets can be extremely complicated and can exhibit dramatic changes on all scales with respect to parameters. In fact the level of complexity is such that their identification or classification is intractable from countable information [20]. However, isolated invariant sets have a continuation property with respect to perturbation of the system [1,21].

The aims of this paper are twofold. First we cast the algebraic structures into a categorical framework in which to develop a sheaf-theoretic description of continuation. Continuation of isolated invariant sets has been the central theme in Conley index theory. Conley [1] and Montgomery [21], and later Salamon [22], formulate continuation in terms of the *space of isolated invariant sets*:

$$\Pi[\mathsf{Isol}] := \Big\{(\phi, S) \;\Big|\; \phi \text{ a flow on } X \text{ and } S \text{ an isolated invariant set for } \phi\Big\}.$$

Two pairs are "close" in the space of isolated invariant sets if the flows are "close" and the isolated invariant sets share a common isolating neighborhood. Two isolated invariant sets are related by continuation if they lie in the same quasicomponent of the space of isolated invariant sets. Both Montgomery and Salamon give proofs of the following crucial result:

Two invariant sets related by continuation share the same Conley index.

Montgomery proves this in the language of étalé spaces (sheaf theory). By choosing an appropriate topology on $\Pi[\mathsf{Isol}]$, the map

$$(\phi, S) \overset{\pi}{\longmapsto} \phi,$$

where the flow ϕ is an element of a topological space of flows on X, is a local homeomorphism. In this paper we expand Montgomery's construction to algebraic structures of global dynamics: lattices of attractors, repellers, and Morse representations. The latter provides an algebraic alternative to the classical approach of labeling Morse sets. Instead, Morse decompositions are formulated as order embeddings [6, Def. 7]. Applying our sheaf-theoretic approach now provides a generalization of Franzosa's theory of continuation of Morse decompositions, cf. [23,24]. By building a categorical framework, we have a natural way to study continuation of these algebraic structures systematically and simultaneously.

Continuation, formulated in terms of étalé spaces for these structures, carries information about bifurcations in parametrized systems. The second aim of this paper is to demonstrate that the sheaf cohomology obtained from these spaces can be used to define new invariants. We investigate sheaf cohomology in the setting of bifurcation theory as an illustration.

For a topological space Λ we define a *parametrized dynamical system* on X as a continuous map $\boldsymbol{\phi}\colon \mathbb{T}\times X\times\Lambda\to X$ such that $\phi^\lambda := \boldsymbol{\phi}(\cdot,\cdot,\lambda)$ is dynamical system on X for all $\lambda\in\Lambda$. The map $\lambda \overset{\boldsymbol{\phi}_*}{\longmapsto} \phi^\lambda$, called the *transpose*, is a continuous map without additional assumptions on the topological spaces Λ and X.

In Section 8.2 we show that the continuation of attractors is conjugacy invariant.

Theorem. *(Conjugacy Invariance Theorem, cf. Theorem 8.7) Let X, Y be compact metric spaces. Suppose $\boldsymbol{\phi}_*\colon \Lambda\to \mathbf{DS}(\mathbb{T},X)$ and $\boldsymbol{\psi}_*\colon \Lambda\to\mathbf{DS}(\mathbb{T},Y)$ are conjugate parametrized dynamical systems. Then, the étalé spaces $\boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Att}]$ and $\boldsymbol{\psi}_*^{-1}\Pi[\mathsf{Att}]$ are homeomorphic.*

Remark 1.1. One can generalize this: a morphism of parametrized dynamical systems on compact Hausdorff spaces yields a morphism of étalé spaces. This requires formulating morphisms of parametrized dynamical systems, which we leave for future work.

In Section 9 continuation sheaves are applied to bifurcations. In terms of sheaf cohomology with respect to the attractor sheaf, $\mathscr{A}^{\boldsymbol{\phi}_*}$, we obtain the following result:

Theorem. *(cf. Theorem 9.14) Let Λ be both contractible and locally contractible, and let $\Lambda'\subset\Lambda$ be a deformation retract of Λ, containing no bifurcation points for $\boldsymbol{\phi}_*$. Suppose that*

$$H^k(\Lambda,\Lambda';\mathscr{A}^{\boldsymbol{\phi}_*})\neq 0,\quad \text{for some}\quad k\geqslant 0.$$

Then, there exists a bifurcation point in $\lambda_0\in\Lambda\smallsetminus\Lambda'$.

In Section 10 attractor sheaf cohomology is computed for the pitchfork, saddle-node, transcritical, and S-shaped bifurcations. As an example, we obtain the following theorem for the pitchfork bifurcation, see Fig. 1.

Theorem. *(cf. Theorem 10.7) Let $\boldsymbol{\phi}_*$ be a parametrized dynamical system over $\mathbb{R}$ with a pitchfork bifurcation at λ_0. Then,*

$$\mathscr{A}^{\boldsymbol{\phi}_*}\quad \text{is acyclic and}\quad H^0(\Lambda;\mathscr{A}^{\boldsymbol{\phi}_*})\cong\mathbb{Z}_2^3.$$

Moreover, there exists a value $\lambda_0\in\mathbb{R}$ such that

$$H^k\big(\Lambda,\Lambda';\mathscr{A}^{\boldsymbol{\phi}_*}\big)\cong\begin{cases}\mathbb{Z}_2^2 & \text{if } k=1 \text{ and } a>\lambda_0;\\ 0 & \text{if } k\neq 1 \text{ or } a\leqslant\lambda_0,\end{cases}$$

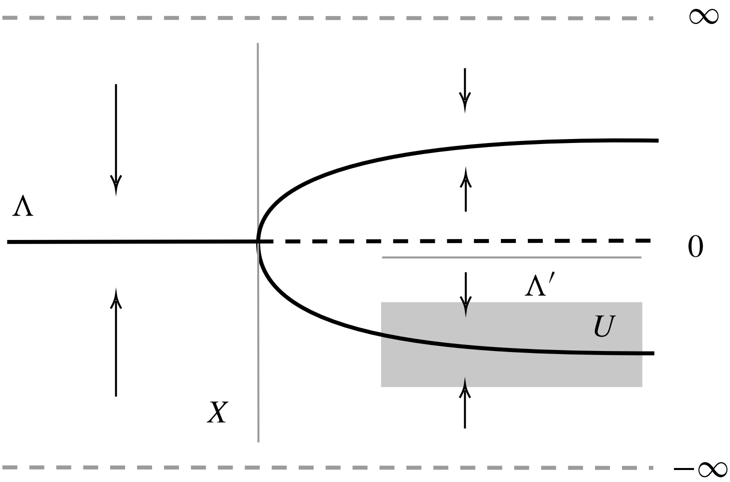


Fig. 1. In the pitchfork bifurcation, the section on $\Lambda' \subset \Lambda$ defined by $\sigma(\lambda) = \big(\lambda, \phi^\lambda, \omega_{\phi^\lambda}(U)\big)$ fails to extend globally. The omega limit set $\omega_{\phi^\lambda}(U)$, cf. Eqn. (2), is the singleton set containing the negative attracting fixed point in this example.

where $\Lambda' = [a, \infty)$. *Furthermore, for* $\Lambda' := (-\infty, a]$, *then* $H^k\big(\Lambda, \Lambda'; \mathscr{A}^{\phi_*}\big) \cong 0$ *for all* k *and for all* $a \in \mathbb{R}$.

Different types of bifurcations can have different cohomology in their attractor sheaves, but if two systems experience the same type of bifurcation, the cohomology is isomorphic. We propose this as a tool for classifying bifurcations, in much the same way singular homology classifies topological spaces. With the strides made in computational dynamics and the success of sheaf theory in topological data analysis, we believe this invariant to be computable by utilizing tools such as the existing theory for combinatorial dynamics [5,8] and cellular sheaf cohomology [25]. This will be the subject of future work.

1.1. Outline

Here with give an overview of the contents the paper. The reader may refer to the table of notation in the Appendix, which also contains additional technical background.

In a series of papers, cf. [2,5,6], we developed an algebraic theory of attractors via distributive lattice theory. We use attractors as the starting point of our approach, which is summarized in Diagram 1.

$$
\begin{array}{ccc}
\mathsf{ANbhd}(\phi) & \overset{c}{\longleftrightarrow} & \mathsf{RNbhd}(\phi) \\
\big\downarrow \omega_\phi & & \big\downarrow \alpha_\phi \\
\mathsf{Att}(\phi) & \overset{*}{\longleftrightarrow} & \mathsf{Rep}(\phi)
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{ANbhd} & \overset{c}{\Longleftrightarrow} & \mathsf{RNbhd} \\
\Big\Downarrow \omega & & \Big\Downarrow \alpha \\
\mathsf{Att} & \overset{*}{\Longleftrightarrow} & \mathsf{Rep}
\end{array}
\qquad
\begin{array}{ccc}
\Pi[\mathsf{ANbhd}] & \overset{\Pi[^c]}{\longleftrightarrow} & \Pi[\mathsf{RNbhd}] \\
\big\downarrow \Pi[\omega] & & \big\downarrow \Pi[\alpha] \\
\Pi[\mathsf{Att}] & \overset{\Pi[^*]}{\longleftrightarrow} & \Pi[\mathsf{Rep}]
\end{array}
\tag{1}
$$

The left diagram, which was established in [2, Diag. (1)], describes the relationship between lattices of attractors and attracting neighborhoods and dually the lattices of repellers and repelling neighborhoods. The ω-limit set and α-limit set operations are lattice homomorphisms. In

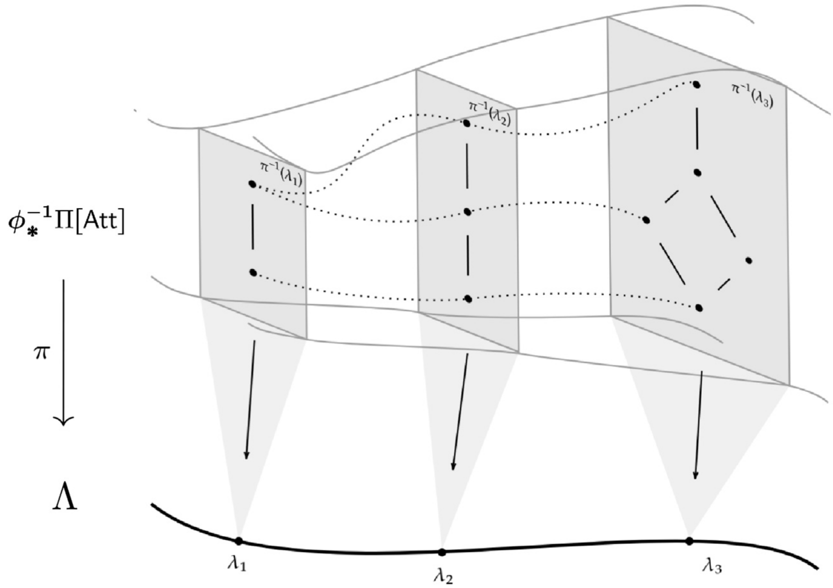


Fig. 2. Given a parametrized dynamical system $\boldsymbol{\phi_*}$, we construct an étalé space of attractors $\boldsymbol{\phi_*^{-1}}\Pi[\mathsf{Att}]$ over parameter space, cf. Sect. 10.1.2. The fiber at a parameter value $\lambda \in \Lambda$ is the attractor lattice of ϕ_λ. Global sections are illustrated by dotted lines. The failure of these global sections to reach all attractors can be measured using sheaf cohomology.

Section 3, we cast these lattices as functors over the category of dynamical systems with ω, α as natural transformations, as shown in the middle diagram. Finally, in Section 5, associated étalé spaces and morphisms are generated as in the right diagram.

In particular, the space $\Pi[\mathsf{Att}]$ is the space of points (ϕ, A), where $A \in \mathsf{Att}(\phi)$ is an attractor, which allows us to define a *sheaf of attractors* over the space of dynamical systems, cf. Fig. 2.

The focus of this paper is to construct and study sheaves which encode the continuation of structures in dynamics. The first seven sections of the paper detail an abstract approach to building sheaves for an arbitrary structure. We routinely return to attractors to showcase how this approach may be applied.

- In Section 2 the set of dynamical systems is equipped with the compact-open topology and then with a categorical structure using the notion of topological conjugacies. This yields the domain category for dynamical structures to be cast as functors and a topology with which to attach algebraic information and sheaves.
- In Section 3 the attracting neighborhood lattice $\mathsf{ANbhd}(\phi)$ and the attractor lattice $\mathsf{Att}(\phi)$ are explicitly expressed as functors from the category of dynamical systems to a category of lattices. These constitute an example to which we apply the later theory.
- In Section 4 prerequisites for continuation are developed: a category with a topology on the objects and a pair of functors, which together we call a *continuation frame.* We prove the existence of an étalé space encoding continuation for these functors. The end of the section constructs morphisms of the étalé spaces using natural transformations between the corresponding functors.
- In Section 5 the framework built in Section 4 is applied to the attractor case from Section 3. This yields the étalé space of attractors. Furthermore, we formulate a morphism of

étalé spaces from the dual repeller operator, seen in Diagram 1. Lastly, we describe the functorial setup for finite sublattices of attractors and Morse representations. This will eventually define a Morse representation sheaf.

- In Section 6 we augment the étalé spaces for attractors, repellers, etc. developed in Section 4 with their algebraic structures.
  We begin by equipping the attractor and repeller étalé spaces with the binary lattice operations. Then, the Conley form on attractors is stated on the level of the attractor étalé space. Later, when we discuss sheaf cohomology, the Conley form will be a crucial tool in building sheaves in an abelian category. The end of Section 6 expands on this, detailing how the algebra of attractors can be stored in a ring.
- In Section 7.2 we use the equivalence between étalé spaces and sheaves. *From an abstract continuation frame we build a sheaf which encodes the continuation of the unstable structure.* The attractor functor begets an *attractor lattice sheaf*, and the Conley form becomes a morphism of sheaves. We also discuss the functors built at the end of Section 6, which give us **Ring**-valued sheaves storing the continuation of attractors. To conclude the section we consider the sheaf of finite attractor sublattices, and the sheaf of Morse representations, as set up in Section 5.

Finally, in Sections 8, 9, and 10 we apply the theory to the setting of parametrized families of dynamical systems, develop the theoretical foundation for studying bifurcation with sheaf cohomology, and compute the sheaf cohomology for some standard, 1-dimensional bifurcations, cf. Fig. 5.

2. Categories of dynamical systems

Throughout this paper we use the following definition of dynamical system. We give spaces of dynamical systems a categorical as well as a topological structure as outlined below.

Definition 2.1. Let $(X, \mathscr{T})$ be a topological space and let $\mathbb{T}$ be the (additive) topological monoid (or group) with topology $\mathscr{T}_{\mathbb{T}}$. A *dynamical system* is a continuous map $\phi\colon \mathbb{T} \times X \to X$ that satisfies

(i) $\phi(0, x) = x$ for all $x \in X$;
(ii) $\phi(t, \phi(s, x)) = \phi(t + s, x)$ for all $s, t \in \mathbb{T}$ and all $x \in X$.

The set of all dynamical systems on the *phase space* X with *time space* $\mathbb{T}$ is denoted by $\mathbf{DS}(\mathbb{T}, X)$. Also, $\phi(t, x)$ may be denoted $\phi_t(x)$.

The time space $\mathbb{T}$ is either $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$, or $\mathbb{R}^+$. In applications, for example those arising from differential equations, it is common for the topology on $\mathbb{T} = \mathbb{R}$ (or $\mathbb{R}^+$) to be the standard topology, which we assume throughout the rest of this paper, but certain results do not require the topology $\mathscr{T}_{\mathbb{T}}$ to have specific properties. Certain properties of the phase space topology $\mathscr{T}$ do play a crucial role. In particular, for clarity of the presentation of the main ideas of this paper, *we always consider the phase space X to be a compact topological space*. For some results, such as Theorem 8.7, we additionally assume a metric on X. Such restrictions are explicitly stated and explained when needed.

We endow $\mathbf{DS}(\mathbb{T}, X)$ with a suitable topology. One natural choice arises by viewing $\mathbf{DS}(\mathbb{T}, X)$ as a function space with the compact-open topology, i.e. the topology generated by the subbasis of sets of the form

$$\left\{\phi \mid \phi(K) \subset U \text{ for } K \text{ compact in } \mathbb{T} \times X \text{ and } U \text{ open in } X\right\}$$

by varying pairs (K, U).

Next we endow $\mathbf{DS}(\mathbb{T}, X)$ with a categorical structure and refer to $\mathbf{DS}(\mathbb{T}, X)$ as the *category of dynamical systems on X over $\mathbb{T}$*. An object $\phi \in \mathsf{ob}(\mathbf{DS}(\mathbb{T}, X))$ is a dynamical system $\phi\colon \mathbb{T} \times X \to X$. A morphism in $\mathsf{hom}(\phi, \psi)$ is defined as $\tau \times h$ such that

(i) $h\colon X \to X$ is a continuous map;
(ii) $\tau\colon \mathbb{T} \times X \to \mathbb{T}$ is a continuous reparametrization that is strictly monotone and bijective for each x and satisfies $\tau(0, x) = 0$;
(iii) the following diagram commutes, cf. [26],

$$\begin{array}{ccc} \mathbb{T} \times X & \xrightarrow{\quad\phi\quad} & X \\ \Big\downarrow{\scriptstyle \tau\times h} & & \Big\downarrow{\scriptstyle h} \\ \mathbb{T} \times X & \xrightarrow{\quad\psi\quad} & X \end{array}$$

We refer to such a morphism $\tau \times h$ as a (topological) quasiconjugacy. Note that $\hom(\phi, \psi)$ can also be endowed with the compact-open topology, so that both the objects and the hom-set of $\mathbf{DS}(\mathbb{T}, X)$ are topological spaces. We abuse notation so that an open subset $\Omega \subset \mathsf{ob}(\mathbf{DS}(\mathbb{T}, X))$ is referred to as an open set Ω in $\mathbf{DS}(\mathbb{T}, X)$.

Remark 2.2. Note that the conditions on reparametrizations imply that $\tau = \mathrm{id}$ in the case that $\mathbb{T} = \mathbb{Z}$, cf. [27, II(7.2)]. This in part motivates the terminology of quasiconjugacy. When $\mathbb{T} = \mathbb{R}$, the case $h = \mathrm{id}$ yields a reparametrization of time $\tau(t, x)$.

Remark 2.3. For special subsets of dynamical systems, such as smooth flows on a manifold, topologies other than the compact-open topology may be more appropriate. *For clarity of presentation, we use the notation* $\mathbf{DS}(\mathbb{T}, X)$ *to mean that the objects and morphisms of this category are given the compact-open topology.* However, in other cases, similar results to those obtained for $\mathbf{DS}(\mathbb{T}, X)$ follow from the abstract theory presented in Section 4.

Remark 2.4. More restrictive choices of the set of morphisms lead to subcategories. For example, one may consider from least restrictive to most restrictive: topological (semi)equivalence with reparametrization of time, topological (semi)conjugacy, or no structure on the morphism set, i.e. $\mathsf{hom}(\phi, \phi) = \{\mathrm{id} \times \mathrm{id}\}$ and $\mathsf{hom}(\phi, \psi) = \emptyset$ when $\phi \neq \psi$, cf. [26,28].

3. Functoriality of dynamics

The study of a dynamical system often focuses on the properties of its invariant sets. A subset $S \subset X$ is *invariant* if it is the union of complete orbits, or equivalently $\phi_t(S) = S$ for all $t \in \mathbb{T}$. One of the most important classes of invariant sets are the attractors. In [2], it is shown that the set of attractors has the algebraic structure of a bounded, distributive lattice. In this section, we characterize such algebraic structures in terms of functors on the category of dynamical systems.

For a given dynamical system $\phi\colon \mathbb{T} \times X \to X$ and a subset $U \subset X$, the *maximal invariant set in* U is

$$\mathrm{Inv}_\phi(U) := \bigcup \{S \subset U : \phi(t, S) = S \text{ for all } t \in \mathbb{T}^+\}.$$

The *omega-limit set* of U is defined by

$$\omega_\phi(U) := \bigcap_{t \geqslant 0} \mathrm{cl} \bigcup_{s \geqslant t} \phi_s(U). \tag{2}$$

Recall from [2] some properties of $\omega_\phi(U)$.

(i) $\omega_\phi(U)$ is compact, closed, and nonempty whenever $U \neq \emptyset$,
(ii) $\omega_\phi(U)$ is an invariant set,[2]
(iii) $\omega_\phi\big(\omega_\phi(U)\big) = \omega_\phi(U)$,
(iv) $\omega_\phi(\mathrm{cl}\, U) = \omega_\phi(U)$,
(v) $\omega_\phi(U \cup V) = \omega_\phi(U) \cup \omega_\phi(V)$.

A subset $U \subset X$ is called an *attracting neighborhood* if $\omega_\phi(U) \subset \mathrm{int}\, U$. Attracting neighborhoods form a bounded, distributive lattice denoted by $\mathsf{ANbhd}(\phi)$. The binary operations are $\cap$ and $\cup$, see [2]. A subset $A \subset X$ is called an *attractor* if there exists an attracting neighborhood $U \subset X$ such that $A = \omega_\phi(U)$, which is a neighborhood of A by definition. Attractors are compact, closed invariant sets, and the set of all attractors is a bounded, distributive lattice $\mathsf{Att}(\phi)$ with binary operations: $A \vee A' = A \cup A'$ and $A \wedge A' := \omega_\phi(A \cap A')$, cf. [2].

Remark 3.1. In the above listed properties of omega-limit sets and attractors, the compactness of X is crucial. If we drop the compactness assumption on X, some of the properties, such as invariance and idempotency, do not hold in general.

When the spaces $\mathbb{T}, X$ are fixed, we often write **DS** in place of $\mathbf{DS}(\mathbb{T}, X)$. The categorical structure of **DS** can now be used to reformulate the above lattices in terms of functors. For notational convenience we write $\psi_t^\dagger := \psi(\tau(t, \cdot), \cdot)$.

Lemma 3.2. *Let $\phi, \psi \in \mathsf{ob}(\mathbf{DS})$ and let $\tau \times h \in \mathsf{hom}(\phi, \psi)$. Then, for all $U \subset Y$ we have*

$$\phi_t(h^{-1}(U)) \subset h^{-1}(\psi_t^\dagger(U)) \quad \forall t \geqslant 0.$$

In particular,

$$\omega_\phi(h^{-1}(U)) = \omega_\phi(h^{-1}(\omega_\psi(U))) \subset h^{-1}(\omega_\psi(U)). \tag{3}$$

Proof. See B. □

[2] For a forward invariant set U, i.e. $\phi_t(U) \subset U$ for all $t \geqslant 0$, it holds that $\mathrm{Inv}_\phi(\mathrm{cl}\, U) = \omega_\phi(U)$.

Now suppose we have $\tau \times h \in \mathsf{hom}(\phi, \psi)$ and $U \in \mathsf{ANbhd}(\psi)$. Then, by Lemma 3.2,

$$\omega_\phi(h^{-1}(U)) \subset h^{-1}(\omega_\psi(U)) \subset h^{-1}(\mathrm{int}\,(U)) \subset \mathrm{int}\,(h^{-1}(U)),$$

where the latter inclusion follows from the continuity of h. Therefore $h^{-1}(U) \in \mathsf{ANbhd}(\phi)$, and the inverse image operator induces a well-defined map $h^{-1} \colon \mathsf{ANbhd}(\psi) \to \mathsf{ANbhd}(\phi)$. This map is in fact a homomorphism by the properties of inverse images, since the lattice operations on $\mathsf{ANbhd}(\phi)$ and $\mathsf{ANbhd}(\psi)$ are union and intersection, so using functor notation, $\mathsf{ANbhd}(\tau \times h) = h^{-1}$. Thus, by assigning to each dynamical system its attracting neighborhood lattice and to each morphism its inverse image operator, by the properties of inverse images and Lemma 3.2, we have a contravariant functor, $\mathsf{ANbhd} \colon \mathbf{DS} \to \mathbf{BDLat}$, from the category of dynamical systems to the category of bounded, distributive lattices.

Remark 3.3. A neighborhood $U \in \mathsf{ANbhd}(\phi)$ is an *attracting block* if $\phi_t(\mathrm{cl}\,U) \subset \mathrm{int}\,(U)$ for all $t > 0$. Now suppose $\tau \times h \in \mathsf{hom}(\phi, \psi)$ and $U \in \mathsf{ABlock}(\psi)$. Then for all $t > 0$

$$\begin{aligned}\phi_t\big(\mathrm{cl}\,(h^{-1}(U))\big) \subset \phi_t\big(h^{-1}(\mathrm{cl}\,U)\big) &\subset h^{-1}\big(\psi_t^\dagger(\mathrm{cl}\,U)\big)\\ &\subset h^{-1}(\mathrm{int}\,(U)) \subset \mathrm{int}\,(h^{-1}(U)),\end{aligned}$$

which implies that $h^{-1}(U) \in \mathsf{ABlock}(\phi)$ so that we can restrict h^{-1} to $h^{-1} \colon \mathsf{ABlock}(\psi) \to \mathsf{ABlock}(\phi)$. As before, $\mathsf{ABlock}(\tau \times h) = h^{-1}$. This makes $\mathsf{ABlock} \colon \mathbf{DS} \to \mathbf{BDLat}$ a contravariant functor. We will primarily use attracting neighborhoods, but Remark 5.4 demonstrates that restricting to attracting blocks does not change the theory.

A similar construction can be used to define a contravariant functor $\mathsf{Att} \colon \mathbf{DS} \to \mathbf{BDLat}$, but its action on morphisms must be modified, since the inverse image of an attractor need not be an attractor.

Proposition 3.4. *Suppose $\tau \times h \in \mathsf{hom}(\phi, \psi)$ and $A \in \mathsf{Att}(\psi)$. Then $\omega_\phi(h^{-1}(A)) \in \mathsf{Att}(\phi)$. Moreover, for $\tau \times h \in \mathsf{hom}(\phi, \psi)$, the map $\omega_\phi \circ h^{-1} \colon \mathsf{Att}(\psi) \to \mathsf{Att}(\phi)$ is a lattice homomorphism.*

Proof. See B. □

Thus, by assigning each dynamical system its attractor lattice and each morphism $\tau \times h \in \mathsf{hom}(\phi, \psi)$ the operator $\mathsf{Att}(\tau \times h) = \omega_\phi \circ h^{-1}$, we have a contravariant functor $\mathsf{Att} \colon \mathbf{DS} \to \mathbf{BDLat}$.

Remark 3.5. If $\tau \times h \in \mathsf{hom}(\phi, \psi)$ is a conjugacy, i.e. $h \colon X \to Y$ is a homeomorphism, then also $\tau^{-1} \times h^{-1} \in \mathsf{hom}(\psi, \phi)$ is a conjugacy, where $\tau^{-1}(s, y)$ is defined by $s = \tau(t, h^{-1}(y))$. As a consequence, $A \in \mathsf{Att}(\phi)$ if and only if $h(A) \in \mathsf{Att}(\psi)$, cf. B.

Remark 3.6. Similar statements as in Lemma 3.2 also hold for α-limit sets, as defined in [4]. Therefore we can define functors $\mathsf{RNbhd}, \mathsf{RBlock} \colon \mathbf{DS} \to \mathbf{BDLat}$ for repelling neighborhoods and repelling blocks analogously. As for Att, one builds $\mathsf{Rep} \colon \mathbf{DS} \to \mathbf{BDLat}$ by replacing ω with α. The details for these constructions are in C.

Remark 3.7. In some situations it is useful to consider attracting neighborhoods in the algebra of regular closed sets using analogous constructions, cf. [5].

Remark 3.8. In the spirit of Montgomery one can also consider the semilattice of isolating neighborhoods $\mathsf{INbhd}(\phi)$ defined by the property $\phi_t(\mathrm{cl}\, U) \subset \mathrm{int}\, U$, cf. [21]. An isolated invariant set is obtained as the maximal invariant set of an isolating neighborhood: $S = \mathrm{Inv}_\phi(U)$. The semilattice of isolated invariant sets is denoted by $\mathsf{Isol}(\phi)$, with $S \wedge S' = \mathrm{Inv}_\phi(S \cap S')$. As for attractors $\mathrm{Inv}_\phi \colon \mathsf{INbhd}(\phi) \to \mathsf{Isol}(\phi)$ is a semilattice homomorphism, and INbhd and Isol may be regarded as functors.

Given this functorial description of dynamical structures, we now turn to the primary focus of this paper, representing continuation of dynamical features in terms of sheaves over $\mathbf{DS}(\mathbb{T}, X)$. To keep the underlying theory flexible, and so as not to repeat theoretical arguments, we first introduce the underlying concepts and theorems abstractly, and then apply this general theory in specific contexts.

4. Abstract continuation

Recall from the introduction that a fundamental feature of Conley theory is that an isolated invariant set continues under perturbation of a dynamical system, which leads to the concept of continuation of isolated invariant sets. In this section, we provide an abstract framework to expand the continuation property to algebraic structures of dynamics.

4.1. *$\mathbf{C}$-structures and categories of elements*

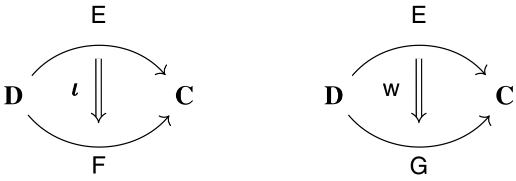
Let $\mathbf{D}$ be a category such that $\mathsf{ob}(\mathbf{D})$ forms a topological space, and let $\mathbf{C}$ be a *concrete* category, i.e. there exists a faithful functor, $\mathbf{C} \to \mathbf{Set}$, into the category of sets. In applications $\mathbf{D}$ is a category of dynamical systems equipped with a topology on $\mathsf{ob}(\mathbf{D})$, such as $\mathbf{DS}(\mathbb{T}, X)$, and $\mathbf{C}$ is the category characterizing the algebraic structure of the dynamical feature to be continued, for example bounded, distributive lattices.

A $\mathbf{C}$-valued contravariant functor on $\mathbf{D}$ is referred to as a $\mathbf{C}$-*structure* on $\mathbf{D}$. Let $\mathsf{E}, \mathsf{G} \colon \mathbf{D} \to \mathbf{C}$ be $\mathbf{C}$-structures and let $\mathsf{w} \colon \mathsf{E} \Rightarrow \mathsf{G}$ be a natural transformation. For objects $\phi \in \mathsf{ob}(\mathbf{D})$ the functors E and G yield objects $\mathsf{E}(\phi)$ and $\mathsf{G}(\phi)$ in $\mathbf{C}$ and the component w_ϕ of the natural transformation yields a morphism $\mathsf{w}_\phi \colon \mathsf{E}(\phi) \to \mathsf{G}(\phi)$.

In applications, typically the functor G represents a dynamical feature such as attractors, and the functor E denotes a corresponding neighborhood feature such as attracting neighborhoods or attracting blocks.

Furthermore, we assume the existence of a constant, contravariant functor $\mathsf{F} \colon \mathbf{D} \to \mathbf{C}$, $\phi \overset{\mathsf{F}}{\mapsto} \mathsf{F}_0$, referred to as the *universe functor*, for which there exists an injective natural transformation $\iota \colon \mathsf{E} \Rightarrow \mathsf{F}$. In dynamics applications, when $\mathbf{D} = \mathbf{DS}(\mathbb{T}, X)$, the universe functor assigns to each ϕ a fixed subalgebra of the Boolean algebra $\mathsf{Set}(X)$, the power set of the phase space, or a fixed subalgebra of the Boolean algebra $\mathscr{R}(X)$, the regular closed subsets of X. For example, if $\mathsf{E} = \mathsf{ANbhd}$, the lattice of attracting neighborhoods, $\mathsf{E}(\phi)$, is a sublattice of $\mathsf{Set}(X)$.

Now we have a span of functors and natural transformations $\mathsf{F} \overset{\iota}{\Leftarrow} \mathsf{E} \overset{\mathsf{w}}{\Rightarrow} \mathsf{G}$ which are summarized in the following diagrams:



(4)

Since **C** is a concrete category, we may regard a functor E into **C** as a **Set**-valued functor, and thus consider $\Pi[\mathsf{E}]$, its *category of elements*, cf. [29,30]. The category of elements construction is used in the next section to generate an étalé space. To define the category of elements $\Pi[\mathsf{E}]$, let $\mathsf{ob}(\Pi[\mathsf{E}])$ be the set of all pairs (ϕ, U) such that $\phi \in \mathsf{ob}(\mathbf{D})$ and $U \in \mathsf{E}(\phi)$. The morphisms of $\Pi[\mathsf{E}]$ are maps $(\phi, U) \to (\phi', U')$ for which there is a **D**-morphism $h\colon \phi \to \phi'$ with $\mathsf{E}(h)(U') = U$. The projection $(\phi, U) \mapsto \phi$ defines a canonical projection functor

$$\boldsymbol{\pi}\colon \Pi[\mathsf{E}] \to \mathbf{D}.$$

Moreover, given a natural transformation between functors, $\mathsf{w}\colon \mathsf{E} \Rightarrow \mathsf{G}$, we have the functor between the associated categories of elements

$$\begin{aligned} \Pi[\mathsf{w}]\colon \Pi[\mathsf{E}] &\to \Pi[\mathsf{G}] \\ (\phi, U) &\mapsto (\phi, \mathsf{w}_\phi(U)). \end{aligned}$$

From the span of functors $\mathsf{F} \overset{\iota}{\Leftarrow} \mathsf{E} \overset{\mathsf{w}}{\Rightarrow} \mathsf{G}$ we obtain a span of functors on the associated categories of elements:

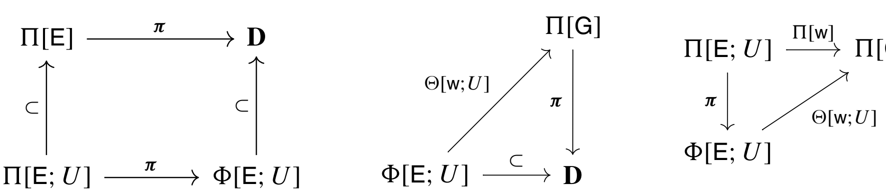
$$\begin{array}{ccccc} \Pi[\mathsf{F}] & \xleftarrow{\Pi[\iota]} & \Pi[\mathsf{E}] & \xrightarrow{\Pi[\mathsf{w}]} & \Pi[\mathsf{G}] \\ (\phi, U) & \longleftarrow\!\shortmid & (\phi, U) & \longmapsto & (\phi, \mathsf{w}_\phi(U)). \end{array} \tag{5}$$

Note that in (5), the set $U \in \mathsf{E}(\phi)$. To localize $\Pi[\mathsf{E}]$, for a fixed element $U \in \mathsf{F}_0$ we define the subcategory $\Pi[\mathsf{E}; U]$ via

$$\mathsf{ob}\big(\Pi[\mathsf{E}; U]\big) := \Big\{(\phi, U) \in \mathsf{ob}(\Pi[\mathsf{E}]) \mid U \in \mathsf{E}(\phi)\Big\}$$

with morphisms $(\phi, U) \to (\phi', U)$ for which there is a **D**-morphism $h\colon \phi \to \phi'$ with $\mathsf{E}(h)(U) = U$.

Applying the projection functor $\boldsymbol{\pi}$ yields a corresponding subcategory $\Phi[\mathsf{E}; U]$ of **D**. The objects of $\Phi[\mathsf{E}; U]$ are given by $\mathsf{ob}\big(\Phi[\mathsf{E}; U]\big) = \big\{\phi \in \mathsf{ob}(\mathbf{D}) \mid U \in \mathsf{E}(\phi)\big\}$ with morphisms $h\colon \phi \to \phi'$ with $\mathsf{E}(h)(U) = U$. This yields the following commutative diagrams:



(6)

where $\Theta[\mathsf{w}; U](\phi) := \big(\phi, \mathsf{w}_\phi(U)\big)$ is called the *partial section functor*, which satisfies:

$$\Theta[\mathsf{w}; U] \circ \boldsymbol{\pi} = \Pi[\mathsf{w}] \quad \text{and} \quad \boldsymbol{\pi} \circ \Theta[\mathsf{w}; U] = \mathsf{id}. \tag{7}$$

We leave it to the reader to verify functoriality.

Remark 4.1. In settings where $\Phi[\mathsf{E}; U]$ is a used as a subset of $\mathbf{D}$ we abuse notation and write $\phi \in \Phi[\mathsf{E}; U]$. The same applies to open subsets $\Omega \subset \mathbf{D}$, cf. Sect. 2.

4.2. Continuation frames and étalé spaces

In the same way that $\mathsf{ob}(\mathbf{D})$ forms topological space, we will equip $\mathsf{ob}(\Pi[\mathsf{G}])$ with a topology. This is done such that the functors $\Theta[\mathsf{w}, U]$ become continuous maps on objects. We will abuse notation and drop the $\mathsf{ob}(-)$ when referring to “elements of $\Pi[\mathsf{G}]$.” A $\mathbf{C}$-structure $\mathsf{E}\colon \mathbf{D} \to \mathbf{C}$ is called *stable* if $\Phi[\mathsf{E}; U]$ is open in $\mathbf{D}$ for all elements $U \in \mathsf{F}_0$. Otherwise a $\mathbf{C}$-structure is said to be *unstable*. In the remainder of the paper we will always assume that $\mathsf{E}\colon \mathbf{D} \to \mathbf{C}$ admits a universe F_0 for which it is stable.

Definition 4.2. A $\mathbf{C}$-*continuation frame* on $\mathbf{D}$ is a triple $(\mathsf{G}, \mathsf{E}, \mathsf{w})$ consisting of $\mathbf{C}$-structures $\mathsf{E}, \mathsf{G}\colon \mathbf{D} \to \mathbf{C}$ and a natural transformation $\mathsf{w}\colon \mathsf{E} \Rightarrow \mathsf{G}$ such that

(i) $\mathsf{w}_\phi\colon \mathsf{E}(\phi) \twoheadrightarrow \mathsf{G}(\phi)$ is surjective for all $\phi \in \mathsf{ob}(\mathbf{D})$;
(ii) E is a stable $\mathbf{C}$-structure.
(iii) The sets $\big\{\phi \in \Phi[\mathsf{E}; U] \cap \Phi[\mathsf{E}; U'] : \mathsf{w}_\phi(U) = \mathsf{w}_\phi(U')\big\}$ are open for all pairs $U, U' \in \mathsf{F}_0$.

Condition (i) can be paraphrased by saying that w is *componentwise* surjective. The $\mathbf{C}$-structure E in a continuation frame is called a *stable extension* of G.

Condition (iii) is crucial for continuity of the sections $\Theta[\mathsf{w}; U]$. In the application of $\mathbf{C}$-structures in dynamics, G is typically unstable as the examples in the next section show. The next step is to topologize $\Pi[\mathsf{G}]$ with the topology generated by

$$\mathscr{B}(\mathsf{G}) := \Big\{\Theta[\mathsf{w}; U](\Omega) \mid U \in \mathsf{F}_0,\ \Omega \subset \Phi[\mathsf{E}; U] \text{ open}\Big\},$$

where $\Theta[\mathsf{w}; U](\Omega)$ is the image under $\Theta[\mathsf{w}; U]$ of objects $\phi \in \Omega$.

Lemma 4.3. *$\mathscr{B}(\mathsf{G})$ is a basis for a topology on $\Pi[\mathsf{G}]$. The maps $\Theta[\mathsf{w}; \mathrm{U}]\colon \Phi[\mathsf{E}; U] \to \Pi[\mathsf{G}]$ are all continuous.*

Proof. Let $\Theta[\mathsf{w}; U](\Omega_1)$, $\Theta[\mathsf{w}; U'](\Omega_2)$ be some elements of $\mathscr{B}(\mathsf{G})$. We can write their intersection in the following way:

$$\Theta[\mathsf{w}; U](\Omega_1) \cap \Theta[\mathsf{w}; U'](\Omega_2) = \Theta[\mathsf{w}; U](\Omega),$$

where we let $\Omega = \Omega_1 \cap \Omega_2 \cap \big\{\phi \in \Phi[\mathsf{E}; U] \cap \Phi[\mathsf{E}; U'] : \mathsf{w}_\phi(U) = \mathsf{w}_\phi(U')\big\}$. For any given $\Theta[\mathsf{w}; U]$ and basis element $\Theta[\mathsf{w}; U'](\Omega)$, one has

$$\Theta[\mathsf{w}; U]^{-1}\big(\Theta[\mathsf{w}; U'](\Omega)\big) = \Omega \cap \big\{\phi \in \Phi[\mathsf{E}; U] \cap \Phi[\mathsf{E}; U'] : \mathsf{w}_\phi(U) = \mathsf{w}_\phi(U')\big\},$$

which is open. □

The functor $\boldsymbol{\pi} : \Pi[\mathsf{G}] \to \mathbf{D}$ may be regarded as a projection $\pi : \mathsf{ob}\big(\Pi[\mathsf{G}]\big) \twoheadrightarrow \mathsf{ob}(\mathbf{D})$, and with the above defined topology on $\mathsf{ob}\big(\Pi[\mathsf{G}]\big)$, it is also a continuous map between topological spaces. In this setting we denote the objects of the category of elements by $\Pi[\mathsf{G}]$, and we show that $(\Pi[\mathsf{G}], \pi)$ is an *étalé space* in the category **Set** by establishing that π is a local homeomorphism, cf. Defn. 7.4.

Theorem 4.4. *Let* $(\mathsf{G}, \mathsf{E}, \mathsf{w})$ *be a* **C***-continuation frame on* **D***. Then, the pair* $(\Pi[\mathsf{G}], \pi)$ *is an étalé space on* **D**.

Proof. To establish $(\Pi[\mathsf{G}], \pi)$ as an étalé space with the above defined topology on $\Pi[\mathsf{G}]$ we show that π is a local homeomorphism.

Let (ϕ, S) be a point in $\Pi[\mathsf{G}]$. Then, since $\mathsf{w}_\phi : \mathsf{E}(\phi) \twoheadrightarrow \mathsf{G}(\phi)$ is surjective for all ϕ, there exists $U \in \mathsf{E}(\phi)$ such that $\mathsf{w}_\phi(U) = S$. Consequently, the point (ϕ, S) is contained in the image of the map $\Theta[\mathsf{w}; U] : \Phi[\mathsf{E}; U] \to \Pi[\mathsf{G}]$, which is *open* by the definition of the topology. The image under π of the set $\mathrm{Im}\big(\Theta[\mathsf{w}; U]\big)$ is the set $\Phi[\mathsf{E}; U]$ which is open by assumption. It remains to show that $\pi : \mathrm{Im}\big(\Theta[\mathsf{w}; U]\big) \to \Phi[\mathsf{E}; U]$ is a homeomorphism.

First we show bijectivity. By definition $\pi : \mathrm{Im}\big(\Theta[\mathsf{w}; U]\big) \to \Phi[\mathsf{E}; U]$ is onto and since $\phi \mapsto (\phi, \mathsf{w}_\phi(U))$ for $\phi \in \Phi[\mathsf{E}; U]$ is a section by Lemma 4.3, we establish bijectivity.

Second we show that $\pi : \mathrm{Im}\big(\Theta[\mathsf{w}; U]\big) \to \Phi[\mathsf{E}; U]$ is continuous and open. Let $\Omega \subset \Phi[\mathsf{E}; U]$ be open. Then, $\pi^{-1}(\Omega) = \Theta[\mathsf{w}; U](\Omega)$ is open by the definition of the topology which proves the continuity of π. Let $\Theta[\mathsf{w}; U'](\Omega)$ be a basic open set. Then,

$$\pi\big(\Theta[\mathsf{w}; U'](\Omega) \cap \mathrm{Im}\big(\Theta[\mathsf{w}; U]\big)\big) = \Omega \cap \big\{\phi \in \Phi[\mathsf{E}; U] \cap \Phi[\mathsf{E}; U'] : \mathsf{w}_\phi(U) = \mathsf{w}_\phi(U')\big\}$$

is open, and thus π is a homeomorphism. This proves that $\Pi[\mathsf{G}]$ is an étalé space in **Set**. □

In the spirit of [21] two points (ϕ, S) and (ϕ', S') are *related by continuation* if they are contained in the same quasicomponent of $\Pi[\mathsf{G}]$, or equivalently (ϕ', S') is contained in the quasicomponent of (ϕ, S). Recall that a quasicomponent of (ϕ, S) of $\Pi[\mathsf{G}]$ is the intersection of all clopen subsets of $\Pi[\mathsf{G}]$ containing (ϕ, S). The following result characterizes this topology.

Proposition 4.5. *Let* $(\mathsf{G}, \mathsf{E}, \mathsf{w})$ *be a* **C***-continuation frame. The topology generated by the basis* $\mathscr{B}(\mathsf{G})$ *is the coarsest topology such that the maps* $\Theta[\mathsf{w}; U] : \Phi[\mathsf{E}; U] \to \Pi[\mathsf{G}]$ *are continuous, and* $\pi : \Pi[\mathsf{G}] \to \mathbf{D}$ *is a local homeomorphism.*

Proof. Theorem 4.4 and Lemma 4.3 give us that $\mathscr{B}(\mathsf{G})$ generates such a topology. Now suppose the maps $\Theta[\mathsf{w}; U] : \Phi[\mathsf{E}; U] \to \Pi[\mathsf{G}]$ are continuous, and $\pi : \Pi[\mathsf{G}] \to \mathbf{D}$ is a local homeomorphism in some topology τ. Let $\Omega \subset \Phi[\mathsf{E}; U]$ be an open set in **D** for some U. We have that $\pi \circ \Theta[\mathsf{w}; U] = \iota$ where ι denotes the inclusion of Ω into **D**. Since ι and π are both local homeomorphisms, so is $\Theta[\mathsf{w}; U]$, which implies that the image $\Theta[\mathsf{w}; U](\Omega)$ is in τ. Every element of $\mathscr{B}(\mathsf{G})$ is of this form; thus, $\mathscr{B}(\mathsf{G})$ is coarser than τ. □

The stable **C**-structure in a **C**-continuation frame yields a second étalé space. Topologize $\Pi[\mathsf{E}]$ as follows. Define the embedding $\Theta[\mathsf{id}; U]\colon\ \Phi[\mathsf{E}; U] \to \Pi[\mathsf{E}]$ as the trivial section $\phi \mapsto (\phi, U)$ and define a subbasis for the topology on $\Pi[\mathsf{E}]$ as follows:

$$\mathscr{B}(\mathsf{E}) := \Big\{\Theta[\mathsf{id}; U](\Omega) \mid U \in \mathsf{F}_0,\ \Omega \subset \Phi[\mathsf{E}; U] \text{ open}\Big\}.$$

Corollary 4.6. *Let* $(\mathsf{G}, \mathsf{E}, \mathsf{w})$ *be a* **C***-continuation frame on* **D***. Then, the pair* $(\Pi[\mathsf{E}], \pi)$ *is an étalé space on* **D**.

Proof. The projection $\pi\colon\ \Pi[\mathsf{E}] \to \mathsf{ob}(\mathbf{D})$ given by $(\phi, U) \mapsto \phi$ is a local homeomorphism with the above defined topology, i.e. $\pi\colon\ \Pi[\mathsf{E}; U] \to \Phi[\mathsf{E}; U]$ is a homeomorphism. □

The category of elements $\Pi[\mathsf{F}]$ trivially gives an étalé space and makes the span of functors in (5) into a span of étalé spaces. A continuous map $\Pi[\mathsf{w}]\colon\ \Pi[\mathsf{E}] \to \Pi[\mathsf{G}]$ is called an *étalé morphism* if the following diagram in commutes, cf. [31, Definition 3.3].

$$\begin{array}{ccc} \Pi[\mathsf{E}] & \xrightarrow{\ \Pi[\mathsf{w}]\ } & \Pi[\mathsf{G}] \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathbf{D} & \end{array} \tag{8}$$

Such a map is then a local homeomorphism by [32, Proposition 2.4.8], [31, Lemma 3.5].

Corollary 4.7. *The map* $\Pi[\mathsf{w}]\colon\ \Pi[\mathsf{E}] \to \Pi[\mathsf{G}]$ *is an étalé morphism.*

Proof. By definition of the topologies on $\Pi[\mathsf{E}]$ and $\Pi[\mathsf{G}]$, the inverse image under $\Pi[\mathsf{w}]$ of a basis element $\Theta[\mathsf{w}; U](\Omega)$, $\Omega \subset \mathbf{D}$ open, is open in $\Pi[\mathsf{E}]$, which proves that $\Pi[\mathsf{w}]$ is continuous. □

4.3. Induced étalé morphisms

The following lemma provides a criterion to construct an étalé space morphism from a natural transformation of structures.

Lemma 4.8. *Let* E, E' *be stable* **C***-structures on* **D** *and let* $\mathsf{n}\colon\ \mathsf{E} \Rightarrow \mathsf{E}'$ *be a natural transformation. Then, the induced functor* $\Pi[\mathsf{n}]\colon\ \Pi[\mathsf{E}] \to \Pi[\mathsf{E}']$*, defined by*

$$(\phi, U) \mapsto (\phi, \mathsf{n}_\phi(U)),$$

defines a morphism of étalé spaces if and only if the sets

$$\{\phi \in \Phi[\mathsf{E}; U] : \mathsf{n}_\phi(U) = U'\},$$

are open for every pair $U, U' \in \mathsf{F}_0$.

Proof. Suppose $\Pi[\mathsf{n}]$ is continuous, and $U, U' \in \mathsf{F}_0$. Then,

$$\left\{\phi \in \Phi[\mathsf{E}; U] \mid \mathsf{n}_\phi(U) = U'\right\} = \pi\left(\Pi[\mathsf{n}]^{-1}\big(\mathrm{Im}(\Theta[\mathsf{id}; U])\big) \cap \mathrm{Im}(\Theta[\mathsf{id}; U'])\right),$$

which is open. Now for the converse. Let $\Theta[\mathsf{id}; U'](\Omega)$ be a subbasis element of $\Pi[\mathsf{E}']$. Then,

$$\bigcup_{U \in \mathsf{F}} \left(\Omega \cap \left\{\phi \in \Phi[\mathsf{E}; U] \mid \mathsf{n}_\phi(U) = U'\right\}\right) = \Pi[\mathsf{n}]^{-1}\big(\Theta[\mathsf{id}; U'](\Omega)\big),$$

which is a union of open sets, and therefore open. □

When the action of n_ϕ is independent of $\phi \in \mathrm{ob}(\mathbf{D})$, the openness condition is trivially satisfied. This condition for stable structures translates to unstable structures in the following proposition.

Proposition 4.9. *Let* $(\mathsf{G}, \mathsf{E}, \mathsf{w})$ *and* $(\mathsf{G}', \mathsf{E}', \mathsf{w}')$ *be continuation frames on* $\mathbf{D}$*, and let* $\mathsf{v}\colon \mathsf{G} \Rightarrow \mathsf{G}'$ *be a natural transformation. Suppose there exists a natural transformation* $\widetilde{\mathsf{v}}\colon \mathsf{E} \Rightarrow \mathsf{E}'$ *such that* $\Pi[\widetilde{\mathsf{v}}]$ *is continuous, and the following diagram commutes:*

$$\begin{array}{ccc} \mathsf{E} & \overset{\widetilde{\mathsf{v}}}{\Longrightarrow} & \mathsf{E}' \\ \Downarrow{\scriptstyle \mathsf{w}} & & \Downarrow{\scriptstyle \mathsf{w}'} \\ \mathsf{G} & \overset{\mathsf{v}}{\Longrightarrow} & \mathsf{G}' \end{array} \tag{9}$$

Then, $\Pi[\mathsf{v}]$ *is a morphism of étalé spaces. The lift* $\widetilde{\mathsf{v}}$ *is called a stable extension of* v.

Proof. We have the following maps on étalé spaces:

$$\begin{array}{ccccc} \Pi[\mathsf{E}] & & \xrightarrow{\quad \Pi[\widetilde{\mathsf{v}}] \quad} & & \Pi[\mathsf{E}'] \\ \downarrow{\scriptstyle \Pi[\mathsf{w}]} & & & & \downarrow{\scriptstyle \Pi[\mathsf{w}']} \\ \Pi[\mathsf{G}] & & \xrightarrow{\quad \Pi[\mathsf{v}] \quad} & & \Pi[\mathsf{G}'] \\ & {\scriptstyle \pi}\searrow & & \swarrow{\scriptstyle \pi} & \\ & & \mathbf{D} & & \end{array}$$

By Corollary 4.7 $\Pi[\mathsf{w}]$ and $\Pi[\mathsf{w}']$ are étalé morphisms and by [32, Proposition 2.4.8], [31, Lemma 3.5] the map $\Pi[\tilde{\mathsf{v}}]$ is an étalé morphism. Diagram chasing then shows that $\Pi[\mathsf{v}]$ is also an étalé morphism by using the same results. □

5. Continuation of attractors and Morse representations

In this section we establish continuation frames for attractors and for finite sublattices of attractors. The latter induces continuation of Morse representations.

5.1. Attractors

In Section 3 we established the functors ANbhd and Att acting between the category of dynamical systems and the category of bounded, distributive lattices. The topologies introduced in Section 2 yield the following result.

Lemma 5.1. ANbhd: $\mathbf{DS}(\mathbb{T}, X) \to \mathbf{BDLat}$ *is a stable structure.*

Proof. As subset of $\mathbf{DS}(\mathbb{T}, X)$ we define $\Phi[\mathsf{ANbhd}; U] := \{\phi \in \mathbf{DS}(\mathbb{T}, X) \mid U \in \mathsf{ANbhd}(\phi)\}$ for any subset $U \subset X$. The condition $U \in \mathsf{ANbhd}(\phi)$ is equivalent to $\omega_\phi(U) \subset \operatorname{int} U$. By [2,3] we have the equivalent characterization: $U \in \mathsf{ANbhd}(\phi)$ if and only if there exists a time $\tau > 0$ such that

$$\phi_t(\operatorname{cl} U) \subset \operatorname{int} U \quad \forall t \geqslant \tau, \tag{10}$$

which is equivalent to

$$\bigcup_{t\in[\tau,2\tau]} \phi_t(\operatorname{cl} U) = \phi\big([\tau, 2\tau] \times \operatorname{cl} U\big) \subset \operatorname{int} U. \tag{11}$$

Indeed, if (10) is satisfied then (11) follows. On the other hand if (11) is satisfied then

$$\begin{aligned}\phi\big([2\tau, 3\tau] \times \operatorname{cl} U\big) &= \bigcup_{t\in[\tau,2\tau]} \phi_{t+\tau}(\operatorname{cl} U) = \phi_\tau\Big(\bigcup_{t\in[\tau,2\tau]} \phi_t(\operatorname{cl} U)\Big)\\ &= \phi_\tau(\operatorname{int} U) \subset \phi_\tau(\operatorname{cl} U) \subset \operatorname{int} U.\end{aligned}$$

By induction $\phi\big([n\tau, (n+1)\tau] \times \operatorname{cl} U\big) \subset \operatorname{int} U$ for all $n \geqslant 1$, which establishes (10). Summarizing,

$$\phi \in \Phi[\mathsf{ANbhd}; U] \qquad \text{if and only} \qquad \phi\big([\tau, 2\tau] \times \operatorname{cl} U\big) \subset \operatorname{int} U \text{ for some } \tau > 0. \tag{12}$$

For any $\tau > 0$ define $K_\tau = [\tau, 2\tau] \times \operatorname{cl} U \subset \mathbb{T} \times X$ which is a compact set. Consider the basic open sets

$$\mathscr{B}\big(K_\tau, \operatorname{int} U\big) := \Big\{\phi \mid \phi(K_\tau) = \phi\big([\tau, 2\tau] \times \operatorname{cl} U\big) \subset \operatorname{int} U\Big\},$$

which are contained in the subbasis for the compact-open topology on $\mathbf{DS}(\mathbb{T}, X)$. By (12) an element $\phi \in \Phi[\mathsf{ANbhd}; U]$ is contained in $\mathscr{B}\big(K_\tau, \operatorname{int} U\big)$ for some $\tau > 0$ and thus $\Phi[\mathsf{ANbhd}; U] \subset \bigcup_{\tau>0} \mathscr{B}\big(K_\tau, \operatorname{int} U\big)$. On the other hand if $\phi \in \mathscr{B}\big(K_\tau, \operatorname{int} U\big)$ for some $\tau > 0$, then (12) implies that $\phi \in \Phi[\mathsf{ANbhd}; U]$ which shows that $\bigcup_{\tau>0} \mathscr{B}\big(K_\tau, \operatorname{int} U\big) \subset \Phi[\mathsf{ANbhd}; U]$ and thus $\Phi[\mathsf{ANbhd}; U] = \bigcup_{\tau>0} \mathscr{B}\big(K_\tau, \operatorname{int} U\big)$ which is a union of basic open set and thus open. □

The next result we prove for isolating neighborhoods, cf. Remark 3.8, which applies to the special case of attracting neighborhoods.

Lemma 5.2. *The sets* $\big\{\phi \in \Phi[\mathsf{INbhd}; U] \cap \Phi[\mathsf{INbhd}; U'] : \operatorname{Inv}_\phi(U) = \operatorname{Inv}_\phi(U')\big\}$ *are open in* $\mathbf{DS}(\mathbb{T}, X)$.

Proof. The proof is identical to Montgomery's proof in [21] for flows. Let $U_1, U_2 \in \mathsf{INbhd}(\phi)$ for $\phi \in \mathbf{DS}(\mathbb{T}, X)$. If $\mathrm{Inv}_\phi(U_1) = \mathrm{Inv}_\phi(U_2) = S$, then $\mathrm{Inv}_\phi(U) = S$ for $U := U_1 \cap U_2$. For any point $x \in V_i := \mathrm{cl}\, U_i \smallsetminus \mathrm{int}\, U$, $i = 1, 2$, there exists a time $\tau > 0$ such that $\phi(\tau, x) \in X \smallsetminus V_i$. By compactness, we may in fact pick $\tau > 0$ such that $\phi(\tau, V_i) \subset X \smallsetminus V_i$. The set

$$\Omega = \Phi[\mathsf{INbhd}; U_1] \cap \Phi[\mathsf{INbhd}; U_2] \cap \{\phi \in \mathbf{DS}(\mathbb{T}, X) : \phi(\tau, V_1) \subset X \smallsetminus V_1\} \\ \cap \{\phi \in \mathbf{DS}(\mathbb{T}, X) : \phi(\tau, V_2) \subset X \smallsetminus V_2\},$$

is open in the compact-open topology. For any $\psi \in \Omega$, $\mathrm{Inv}_\psi(U_1) = \mathrm{Inv}_\psi(U) = \mathrm{Inv}_\psi(U_2)$, so we are done. $\square$

In particular, $\mathrm{Inv}_\phi(U) = \omega_\phi(U)$ when $U \in \mathsf{ANbhd}(\phi)$ by Corollary 3.6 of [2]. Consequently, the triple $\big(\mathsf{Att}, \mathsf{ANbhd}, \omega\big)$ is a **BDLat**-continuation frame on $\mathbf{DS}(\mathbb{T}, X)$, and ANbhd is a stable extension for Att. By Theorem 4.4 we have that $(\Pi[\mathsf{Att}], \pi)$ is an étalé space in **Set**.

Remark 5.3. Similar to attractors, the triple $(\mathsf{INbhd}, \mathsf{Isol}, \mathrm{Inv})$ is a **MLat**-continuation frame on $\mathbf{DS}(\mathbb{T}, X)$, where **MLat** is the category of bounded, meet-semilattices, cf. [21].

Remark 5.4. Stable extensions in a continuation frame are not unique. Following Remark 3.3, attracting blocks define attractors via $\omega_\phi \colon \mathsf{ABlock}(\phi) \twoheadrightarrow \mathsf{Att}(\phi)$. As before we may regard $\mathsf{ABlock} \colon \mathbf{DS}(\mathbb{T}, X) \to \mathbf{BDLat}$ as a contravariant functor which is a stable extension of $\mathsf{Att} \colon \mathbf{DS}(\mathbb{T}, X) \to \mathbf{BDLat}$. Using the inclusion transformation $\iota \colon \mathsf{ABlock} \Rightarrow \mathsf{ANbhd}$, we obtain the following commutative diagram of transformations:

$$\begin{array}{ccc} \mathsf{ABlock} & \overset{\iota}{\Longrightarrow} & \mathsf{ANbhd} \\ \Downarrow \omega & & \Downarrow \omega \\ \mathsf{Att} & \overset{\mathrm{id}}{\Longrightarrow} & \mathsf{Att} \end{array}$$

Proposition 4.9 obtains an isomorphism between the étalé spaces generated from the two continuation frames $(\mathsf{Att}, \mathsf{ANbhd}, \omega)$ and $(\mathsf{Att}, \mathsf{ABlock}, \omega)$.

The functor Att is the structure we wish to continue, with stable extension $\omega \colon \mathsf{ANbhd} \Rightarrow \mathsf{Att}$. This gives

$$\mathbf{DS}(\mathbb{T}, X) \ \underset{\mathsf{Att}}{\overset{\mathsf{ANbhd}}{\rightrightarrows}}\ \mathbf{BDLat} \quad (\omega \colon \mathsf{ANbhd} \Downarrow \mathsf{Att}) \qquad\qquad \begin{array}{ccc} & & \Pi[\mathsf{Att}] \\ & \overset{\Theta[\omega; U]}{\nearrow} & \downarrow \pi \\ \Phi[\mathsf{ANbhd}; U] & \overset{\subset}{\longrightarrow} & \mathbf{DS}(\mathbb{T}, X) \end{array}$$

We have the partial section map

$$\Theta[\omega; U] \colon \Phi[\mathsf{ANbhd}; U] \to \Pi[\mathsf{Att}],$$

which maps a dynamical system ϕ with attracting neighborhood U to the pair (ϕ, A) with its associated attractor $A = \omega_\phi(U)$. Since ω_ϕ is surjective, given a pair $(\phi, A) \in \Pi[\mathsf{Att}]$, there exists an attracting neighborhood U such that $\Theta[\omega; U](\phi) = (\phi, A)$.

Remark 5.5. Following Remark 3.6, we can build a continuation frame $(\mathsf{Rep}, \mathsf{RNbhd}, \alpha)$ for repellers, which gives us an étalé space $\Pi[\mathsf{Rep}]$. Proposition 4.9 and the setup in C allow us to construct an isomorphism of étalé spaces from the dual repeller operator

$$\Pi[*]\colon \Pi[\mathsf{Att}] \to \Pi[\mathsf{Rep}] \quad (\phi, A) \mapsto (\phi, A^*),$$

by using the set complement on attracting neighborhoods as a stable extension of $*$. To view set complement and $*$ as natural transformations, one can augment the hom-set of **BDLat** with anti-homomorphisms or compose with an opposite functor. This technicality appears again in 5.2 with $\boldsymbol{\mu}$ and $\boldsymbol{\tau}$. Alas, note that so far these are **Set**-valued étalé spaces. When we introduce lattice operations in Section 6, this will become an anti-isomorphism of **BDLat**-valued étalé spaces.

Remark 5.6. Note that the dual repeller operator $*$ is dependent on the underlying system ϕ:

$$A \mapsto A^* = \{x \in X : \omega_\phi(x) \cap A = \emptyset\}. \tag{13}$$

For convenience of notation, we will omit the subscript when the underlying system is understood.

5.2. *Morse representations*

Define the set $\mathrm{sub}_{\mathrm{F}}\mathsf{Att}(\phi)$ consisting of all the finite sublattices $\mathsf{A} \subset \mathsf{Att}(\phi)$. Every finite sublattice is understood to contain at least the elements $\emptyset$ and $\omega_\phi(X)$. The set of finite sublattices can be given the structure of a semibounded lattice

$$\mathsf{A} \wedge \mathsf{A}' := \mathsf{A} \cap \mathsf{A}', \quad \mathsf{A} \vee \mathsf{A}' := [\mathsf{A} \cup \mathsf{A}'], \quad \mathsf{A}, \mathsf{A}' \subset \mathsf{Att}(\phi),$$

where $[\mathsf{A} \cup \mathsf{A}']$ is the smallest sublattice containing $\mathsf{A} \cup \mathsf{A}'$. Note that, since the $\mathsf{Att}(\phi)$ may be infinite, there may be no maximal element in $\mathrm{sub}_{\mathrm{F}}\mathsf{Att}(\phi)$, and hence $\mathrm{sub}_{\mathrm{F}}\mathsf{Att}(\phi)$ may not be a bounded lattice. The lattice $\mathrm{sub}_{\mathrm{F}}\mathsf{Att}(\phi)$ has minimal element $\{\emptyset, \omega_\phi(X)\}$. Also $\mathrm{sub}_{\mathrm{F}}\mathsf{Att}(\phi)$ is not a distributive lattice in general. The assignment $\mathsf{Att}(\phi) \to \mathrm{sub}_{\mathrm{F}}\mathsf{Att}(\phi)$ may be regarded as covariant functor $\mathrm{sub}_{\mathrm{F}}\colon \mathbf{BDLat} \to \mathbf{Lat}$, where **Lat** is the category of lattices. Indeed, if L and K are bounded, distributive lattices and $g\colon \mathsf{L} \to \mathsf{K}$ is a lattice homomorphism (preserves 0 and 1), then the inclusion of a finite sublattice $i\colon \mathsf{L}' \subset \mathsf{L}$ defines a finite sublattice $\mathsf{K}' \subset \mathsf{K}$ as the range of the composition $g \circ i$. We define the arrow

$$\mathrm{sub}_{\mathrm{F}}(g)\colon \mathrm{sub}_{\mathrm{F}}\mathsf{L} \to \mathrm{sub}_{\mathrm{F}}\mathsf{K}, \quad \mathsf{L}' \mapsto \mathrm{sub}_{\mathrm{F}}(g)(\mathsf{L}') := \mathsf{K}'.$$

The composition of functors yields the contravariant functors $\mathrm{sub}_{\mathrm{F}} \circ \mathsf{ANbhd}$ and $\mathrm{sub}_{\mathrm{F}} \circ \mathsf{Att}$ which provide the following diagrams:

$$\mathbf{DS}(\mathbb{T}, X) \underset{\mathrm{sub_F}\mathsf{Set}}{\overset{\mathrm{sub_F}\mathsf{ANbhd}}{\underset{\Downarrow\,\boldsymbol{\iota}}{\rightrightarrows}}} \mathbf{Lat} \qquad \mathbf{DS}(\mathbb{T}, X) \underset{\mathrm{sub_F}\mathsf{Att}}{\overset{\mathrm{sub_F}\mathsf{ANbhd}}{\underset{\Downarrow\,\boldsymbol{\omega}}{\rightrightarrows}}} \mathbf{Lat} \tag{14}$$

where $\boldsymbol{\iota}$ is the natural transformation defined by inclusion. The bounded lattice $\mathrm{sub_F}\mathsf{Set}(X)$ consists of finite rings of sets over X in the universe for the continuation frame we construct. Moreover, $\boldsymbol{\omega}$ is a natural transformation defined as follows.

Let $\mathsf{U} \in \mathrm{sub_F}\mathsf{ANbhd}(\phi)$, then $\boldsymbol{\omega}_\phi(\mathsf{U}) := \{A = \omega_\phi(U) \mid U \in \mathsf{U}\}$. This construction also yields the lattice homomorphism $\omega_\phi \colon \mathsf{U} \twoheadrightarrow \boldsymbol{\omega}_\phi(\mathsf{U})$. From the definition of $\mathrm{sub_F}\mathsf{ANbhd}$ we obtain the following lemma.

Lemma 5.7. $\mathrm{sub_F}\mathsf{ANbhd}\colon \mathbf{DS}(\mathbb{T}, X) \to \mathbf{Lat}$ *is a stable structure. The sets* $\{\phi \in \mathbf{DS}(\mathbb{T}, X) : \boldsymbol{\omega}_\phi(\mathsf{U}) = \boldsymbol{\omega}_\phi(\mathsf{U}')\}$ *are open.*

Proof. For any finite sublattice $\mathsf{U} \in \mathrm{sub_F}\mathsf{Set}(X)$ we observe that

$$\Phi[\mathrm{sub_F}\mathsf{ANbhd}; \mathsf{U}] = \bigcap_{U \in \mathsf{U}} \Phi[\mathsf{ANbhd}; U],$$

which, by Lemma 5.1, is open. Suppose we have two finite sublattices $\mathsf{U}, \mathsf{U}' \in \mathrm{sub_F}\mathsf{ANbhd}(\psi)$ such that $\boldsymbol{\omega}_\psi(\mathsf{U}) = \boldsymbol{\omega}_\psi(\mathsf{U}')$. Then for each $U \in \mathsf{U}$, there exists a $U' \in \mathsf{U}'$ such that $\omega_\psi(U) = \omega_\psi(U')$, and vice versa. Take a finite intersection of open sets

$$\bigcap_{\substack{U \in \mathsf{U}, U' \in \mathsf{U}' \\ \omega_\psi(U) = \omega_\psi(U')}} \{\phi \in \Phi[\mathsf{ANbhd}; U] \cap \Phi[\mathsf{ANbhd}; U'] : \omega_\phi(U) = \omega_\phi(U')\}$$
$$\subset \{\phi \in \Phi[\mathrm{sub_F}\mathsf{ANbhd}; \mathsf{U}] \cap \Phi[\mathrm{sub_F}\mathsf{ANbhd}; \mathsf{U}'] : \boldsymbol{\omega}_\phi(\mathsf{U}) = \boldsymbol{\omega}_\phi(\mathsf{U}')\},$$

and we are done. $\square$

By construction the natural transformation $\boldsymbol{\omega}\colon \mathrm{sub_F}\mathsf{ANbhd} \Longrightarrow \mathrm{sub_F}\mathsf{Att}$ is surjective, which in combination with Lemma 5.7 implies the following:

Lemma 5.8. *The triple* $(\mathrm{sub_F}\mathsf{Att}, \mathrm{sub_F}\mathsf{ANbhd}, \boldsymbol{\omega})$ *is a* **Lat***-continuation frame.*

Following [6,7] we associate an ordered partition $\mathsf{T}(\mathsf{U})$ for every finite sublattice $\mathsf{U} \in \mathrm{sub_F}\mathsf{ANbhd}(\phi)$: let $\mathsf{J}(\mathsf{U})$ be the poset of join-irreducible elements in U, elements U with a unique predecessor $\overleftarrow{U}$, and for $U \in \mathsf{J}(\mathsf{U})$ define $T(U) = U \smallsetminus \overleftarrow{U}$. The poset $\mathsf{J}(\mathsf{U})$, ordered by inclusion, induces an isomorphic poset structure on $\mathsf{T}(\mathsf{U}) := \{T(U) \mid U \in \mathsf{J}(\mathsf{U})\}$ with $T(U) \leqslant T(U')$ if and only if $U \subseteq U'$. The poset $\mathsf{T}(\mathsf{U})$ is called a *Morse tessellation* for ϕ. We can similarly consider a finite sublattice $\mathsf{A} \in \mathrm{sub_F}\mathsf{Att}(\phi)$, and let $\mathsf{J}(\mathsf{A})$ be the poset of join-irreducible elements in A. Define a map $\mathsf{J}(\mathsf{A}) \to \mathsf{Invset}(\phi)$ by $A \mapsto M(A) := A \cap (\overleftarrow{A})^* = C_{\mathsf{Att}}(A, \overleftarrow{A})$. The elements $M(A)$

compose the isomorphic poset $\mathsf{M}(\mathsf{A}) := \{M(A) \mid A \in \mathsf{J}(\mathsf{A})\}$ with $M(A) \leqslant M(A')$ if and only if $A \subseteq A'$. The poset $\mathsf{M}(\mathsf{A})$ is called a *Morse representation* for ϕ.

Let $\mathsf{MRepr}(\phi)$ denote the set of Morse representations for a dynamical system ϕ, $\mathsf{MTess}(\phi)$ the set of Morse tessellations, and $\mathsf{O}\colon \mathbf{Poset} \to \mathbf{Lat}$ the downset functor, so that $\mathsf{O}(\mathsf{T})$ is the lattice of downsets of T. Then, there are bijections:

$$\boldsymbol{\tau}_\phi\colon \mathrm{sub}_{\mathrm{F}}\mathsf{ANbhd}(\phi) \to \mathsf{MTess}(\phi), \quad \boldsymbol{\tau}_\phi(\mathsf{U}) := \mathsf{T}(\mathsf{U}), \;\; \boldsymbol{\tau}_\phi^{-1}(\mathsf{T}) := \Big\{U = \bigcup I \mid I \in \mathsf{O}(\mathsf{T})\Big\}.$$

$$\boldsymbol{\mu}_\phi\colon \mathrm{sub}_{\mathrm{F}}\mathsf{Att}(\phi) \to \mathsf{MRepr}(\phi), \quad \boldsymbol{\mu}_\phi(\mathsf{A}) := \mathsf{M}(\mathsf{A}), \;\; \boldsymbol{\mu}_\phi^{-1}(\mathsf{M}) := \Big\{A = \bigcup_{M \in I} W^u(M) \mid I \in \mathsf{O}(\mathsf{T})\Big\}.$$

These bijections let $\mathsf{MRepr}(\phi)$ and $\mathsf{MTess}(\phi)$ inherit the lattice structure of their dual counterparts $\mathrm{sub}_{\mathrm{F}}\mathsf{Att}(\phi)$ and $\mathrm{sub}_{\mathrm{F}}\mathsf{ANbhd}(\phi)$ respectively:

$$\mathsf{T} \vee \mathsf{T}' := \boldsymbol{\tau}_\phi\big(\boldsymbol{\tau}_\phi^{-1}(\mathsf{T}) \wedge \boldsymbol{\tau}_\phi^{-1}(\mathsf{T}')\big), \quad \mathsf{T} \wedge \mathsf{T}' := \boldsymbol{\tau}_\phi\big(\boldsymbol{\tau}_\phi^{-1}(\mathsf{T}) \vee \boldsymbol{\tau}_\phi^{-1}(\mathsf{T}')\big)$$
$$\mathsf{M} \vee \mathsf{M}' := \boldsymbol{\mu}_\phi\big(\boldsymbol{\mu}_\phi^{-1}(\mathsf{M}) \wedge \boldsymbol{\mu}_\phi^{-1}(\mathsf{M}')\big), \quad \mathsf{M} \wedge \mathsf{M}' := \boldsymbol{\mu}_\phi\big(\boldsymbol{\mu}_\phi^{-1}(\mathsf{M}) \vee \boldsymbol{\mu}_\phi^{-1}(\mathsf{M}')\big).$$

As such, $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ become lattice isomorphisms. We can view MRepr and MTess as functors assigning dynamical systems their Morse representations and Morse tessellations respectively. Define an action on morphisms using $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$:

$$h \in \hom(\phi, \psi), \quad \mathsf{MTess}(h) := \boldsymbol{\tau}^{-1} \circ \mathrm{sub}_{\mathrm{F}}\mathsf{ANbhd}(h) \circ \boldsymbol{\tau}, \quad \mathsf{MRepr}(h) := \boldsymbol{\mu}^{-1} \circ \mathrm{sub}_{\mathrm{F}}\mathsf{Att}(h) \circ \boldsymbol{\mu}.$$

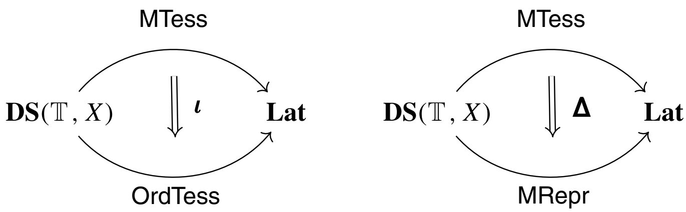
$\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ become natural transformations in this way. The result is the following diagram of functors:

$$\mathbf{DS}(\mathbb{T}, X) \underset{\mathsf{MTess}}{\overset{\mathrm{sub}_{\mathrm{F}}\mathsf{ANbhd}}{\rightrightarrows}} \mathbf{Lat} \;\; (\boldsymbol{\tau} \Updownarrow \boldsymbol{\tau}^{-1}) \qquad \mathbf{DS}(\mathbb{T}, X) \underset{\mathsf{MRepr}}{\overset{\mathrm{sub}_{\mathrm{F}}\mathsf{Att}}{\rightrightarrows}} \mathbf{Lat} \;\; (\boldsymbol{\mu} \Updownarrow \boldsymbol{\mu}^{-1}) \tag{15}$$

Let $\boldsymbol{\Delta}$ be the natural transformation defined by

$$\boldsymbol{\Delta} := \boldsymbol{\mu} \circ \boldsymbol{\omega} \circ \boldsymbol{\tau}^{-1}.$$

Given a Morse tessellation T we obtain a Morse representation via $\mathsf{T} \mapsto \boldsymbol{\Delta}_\phi(\mathsf{T}) =: \mathsf{M}$. By the above correspondences $\mathsf{M} = \boldsymbol{\mu}_\phi(\mathsf{A})$ with $\mathsf{A} = \boldsymbol{\mu}_\phi^{-1}(\boldsymbol{\Delta}_\phi(\mathsf{T}))$ and $\mathsf{T} = \boldsymbol{\tau}_\phi(\mathsf{U})$ with $\mathsf{U} = \boldsymbol{\tau}_\phi^{-1}(\mathsf{T})$ and $\boldsymbol{\omega}_\phi(\boldsymbol{\tau}_\phi^{-1}(\mathsf{T})) = \mathsf{A}$. For the elements U and $\boldsymbol{\omega}_\phi(\mathsf{U})$ we have the homomorphism $\omega_\phi\colon \mathsf{U} \twoheadrightarrow \mathsf{A} = \boldsymbol{\omega}_\phi(\mathsf{U})$. This implies the following relation for Morse tessellations and Morse representations. Associated with U we have $\mathsf{T} = \boldsymbol{\tau}_\phi(\mathsf{U})$ maps to $\mathsf{M} = \boldsymbol{\Delta}_\phi(\mathsf{T})$ and we obtain a canonical embedding $\iota\colon \mathsf{M} \hookrightarrow \mathsf{T}$, which will be called a *tessellated Morse decomposition*. The embedding ι is induced by the homomorphism ω_ϕ. The continuation frame is given by the following diagrams:



(16)

and establishes the continuation frame $\big(\mathsf{MRepr}, \mathsf{MTess}, \boldsymbol{\Delta}\big)$. Here the universe is given by $\mathsf{OrdTess}(X)$ which is the (complete) bounded lattice of finite ordered tessellations of X.

Remark 5.9. The lattice operations on $\mathsf{MRepr}(\phi)$ and $\mathsf{MTess}(\phi)$ are motivated by the duality between (Priestley) pre-orders and sublattices, i.e. for a bounded distributive lattice L there exist an anti-isomorphism to the lattice of Priestley pre-order on the Priestley $\Sigma\mathsf{L}$ and the lattice sub L of sublattices of L, cf. [33, Thm. 3.7], [34, Thm. 2.5].

6. Algebraic constructions

In this section we incorporate the binary operations of lattices, groups, rings, etc. and augment the étalé spaces with these operations.

6.1. Binary operations and lattices

Given two étalé spaces (Π, π), (Π', π') over a topological space. Define

$$\Pi \bullet \Pi' := \big\{(\sigma, \sigma') \in \Pi \times \Pi' : \pi(\sigma) = \pi'(\sigma')\big\},$$

which is also an étalé space with the same projection map and the product topology, cf. [31, Sect. 2.5].

Proposition 6.1. *Suppose the category* $\mathbf{C}$ *has concrete binary products.*[3] *Let* $(\mathsf{G}, \mathsf{E}, \mathsf{w})$, $(\mathsf{G}', \mathsf{E}', \mathsf{w}')$ *be* $\mathbf{C}$*-continuation frames on* $\mathbf{D}$*. Then,* $(\mathsf{G} \times \mathsf{G}', \mathsf{E} \times \mathsf{E}', \mathsf{w} \times \mathsf{w}')$ *is a continuation frame and the map*

$$g\colon \Pi[\mathsf{G} \times \mathsf{G}'] \to \Pi[\mathsf{G}] \bullet \Pi[\mathsf{G}'], \qquad \big(\phi, (S, S')\big) \mapsto \big((\phi, S), (\phi, S')\big)$$

is a homeomorphism.

Proof. Since both w and w' are surjective componentwise, so is $\mathsf{w} \times \mathsf{w}'$. For the openness conditions:

$$\Phi[\mathsf{G} \times \mathsf{G}'; (U, U')] = \big\{\phi \in \mathrm{ob}(\mathbf{D}) \mid (U, U') \in (\mathsf{G} \times \mathsf{G}')(\phi)\big\} = \Phi[\mathsf{G}; U] \cap \Phi[\mathsf{G}'; U'].$$

$$\begin{aligned}\{\phi \in \mathbf{D} : (\mathsf{w} \times \mathsf{w}')_\phi(U_1, U_2) = (\mathsf{w} \times \mathsf{w}')_\phi(U_1', U_2')\} =& \{\phi \in \mathbf{D} : \mathsf{w}_\phi(U_1) = \mathsf{w}_\phi(U_1')\} \\ &\cap \{\phi \in \mathbf{D} : \mathsf{w}'_\phi(U_2) = \mathsf{w}'_\phi(U_2')\}\end{aligned}$$

[3] The product in $\mathbf{C}$ is the product of sets after applying the forgetful functor.

which is open. Bijectivity of g is immediate; it remains to be shown that g is continuous and open on subbasis elements. Let $U, U' \in \mathsf{F}$ and Ω, Ω' open in $\Phi[\mathsf{E}; U]$ and $\Phi[\mathsf{E}'; U']$ respectively.

Then,

$$g^{-1}\Big(\Theta[\mathsf{w}; U](\Omega) \times \Theta[\mathsf{w}; U'](\Omega') \cap \Pi[\mathsf{G}] \bullet \Pi[\mathsf{G}']\Big) = \Theta\big[\mathsf{w} \times \mathsf{w}; (U, U')\big](\Omega \cap \Omega'),$$

which is open. Similarly, letting $U, U' \in \mathsf{F}$ and $\Omega \subset \Phi[\mathsf{E} \times \mathsf{E}'; (U, U')]$ open we have:

$$g\Big(\Theta[\mathsf{w} \times \mathsf{w}'; (U, U')]\Big) = \Theta[\mathsf{w}; U](\Omega) \times \Theta[\mathsf{w}', U'](\Omega) \cap \Pi[\mathsf{G}] \bullet \Pi[\mathsf{G}']$$

which proves that g is an open map and therefore a homeomorphism. □

Remark 6.2. The universe functor in the product continuation frame is the product $\mathsf{F} \times \mathsf{F}'$.

We can apply Propositions 4.9 and 6.1 to the **BDLat**-continuation frames $(\mathsf{Att}, \mathsf{ANbhd}, \omega)$ and $(\mathsf{Rep}, \mathsf{RNbhd}, \alpha)$ to interpret lattice operations as morphisms of étalé spaces. This permits us to regard $\Pi[\mathsf{Att}]$ and $\Pi[\mathsf{Rep}]$ as **BDLat**-valued.

For example $\wedge_\phi \colon \mathsf{Att}(\phi) \times \mathsf{Att}(\phi) \to \mathsf{Att}(\phi)$ given by $(A, A') \mapsto A \wedge A'$ forms a natural transformation

$$\wedge \colon \mathsf{Att} \times \mathsf{Att} \Rightarrow \mathsf{Att}.$$

From Proposition 4.9 $\widetilde{\wedge} \colon \mathsf{ANbhd} \times \mathsf{ANbhd} \Rightarrow \mathsf{ANbhd}$, given by $(U, U') \mapsto U \cap U'$ with $\omega_\phi(U) = A$ and $\omega_\phi(U') = A'$, acts as a lift for $\wedge$ which yields an étalé space morphism from $\Pi[\mathsf{Att} \times \mathsf{Att}]$ to $\Pi[\mathsf{Att}]$. Combining the latter with Proposition 6.1 yields an étalé space morphism:

$$\Pi[\wedge] \colon \Pi[\mathsf{Att}] \bullet \Pi[\mathsf{Att}] \to \Pi[\mathsf{Att}], \quad \big((\phi, A), (\phi, A')\big) \mapsto (\phi, A \wedge A'),$$

which establishes $\wedge$ as a continuous binary operation on $\Pi[\mathsf{Att}]$. The same can be achieved for $\vee$. Absorption, distributivity, and associativity follows immediately from the properties of $\wedge$ and $\vee$. It remains to show that the assignments of the neutral elements

$$\phi \mapsto (\phi, \emptyset) \in \Pi[\mathsf{Att}], \qquad \phi \mapsto \big(\phi, \omega_\phi(X)\big) \in \Pi[\mathsf{Att}],$$

are continuous. By composing the constant sections $\Theta[\mathsf{id}; \emptyset], \Theta[\mathsf{id}; X] \colon \mathbf{DS}(\mathbb{T}, X) \to \Pi[\mathsf{ANbhd}]$ with the continuous map $\Pi[\omega]$ we obtain the desired result. A similar argument holds for Rep. Consequently, we have established $\Pi[\mathsf{Att}]$ and $\Pi[\mathsf{Rep}]$ as **BDLat**-valued étalé spaces. We later explore abelian structures and ring structures which are used in the treatment of sheaf cohomology.

6.2. The Conley form

Recall that the Conley form assigns to two attractors $A, A' \in \mathsf{Att}(\phi)$ an associated invariant set $(A, A') \mapsto \mathsf{C}_{\mathsf{Att}}(A, A') := A \cap A'^*$, where $A'^* \in \mathsf{Rep}(\phi)$ is dual to A' in the sense that $A'^* = \alpha_\phi(U^c)$ where $U \in \mathsf{ANbhd}(\phi)$ with $\omega_\phi(U) = A'$. The repeller A'^* is called the *dual repeller* to A'. The Conley form has a universal property in the sense that it is a unique extension of set-difference for bounded, distributive lattices, cf. [6].

A *Morse neighborhood* is a subset $T \subset X$ given by $T = U \cap V$ with $U \in \mathsf{ANbhd}(\phi)$ and $V \in \mathsf{RNbhd}(\phi)$. It holds that $\mathrm{Inv}_\phi(T) = \omega_\phi(U) \cap \alpha_\phi(V) := M$ which is called a *Morse set*. By construction $M \subset \mathrm{int}\, T$, cf. [6]. The Morse sets are denoted by $\mathsf{Morse}(\phi)$ which is a bounded, meet-semilattice with binary operation $M \wedge M' := \mathrm{Inv}_\phi(M \cap M')$. The Morse neighborhoods are denoted by $\mathsf{MNbhd}(\phi)$ and form a bounded, meet-semilattice with intersection as binary operation. Both $\emptyset$ and $\omega_\phi(X)$ are neutral elements. As before, $\mathrm{Inv}\colon \mathsf{MNbhd} \Rightarrow \mathsf{Morse}$ is a stable **MLat**-structure, where **MLat** is the category of bounded, meet-semilattices. The triple $(\mathsf{Morse}, \mathsf{MNbhd}, \mathrm{Inv})$ is a **MLat**-continuation frame and by the general theory in Section 4 we obtain the **MLat**-étalé space $\Pi[\mathsf{Morse}]$ of Morse sets.

By the same token we can treat the Conley form as natural transformation

$$\mathsf{C}_{\mathsf{Att}}\colon \mathsf{Att} \times \mathsf{Att} \Longrightarrow \mathsf{Morse},$$

where the functor Morse assigns the bounded, meet-semilattice of Morse sets to ϕ. By Proposition 6.1 this leads to a continuous operation

$$\Pi[\mathsf{C}_{\mathsf{Att}}]\colon \Pi[\mathsf{Att}] \bullet \Pi[\mathsf{Att}] \to \Pi[\mathsf{Morse}] \quad \big((\phi, A)(\phi, A')\big) \mapsto \big(\phi, \mathsf{C}_{\mathsf{Att}}(A, A')\big).$$

The map $\Pi[\mathsf{C}_{\mathsf{Att}}]$ will play a role in setting up the appropriate algebraic construction for sheaf cohomology.

A variation on the Conley form is the *symmetric Conley form* which is defined as follows:

$$(A, A') \mapsto \mathsf{C}^*(A, A') := \mathsf{C}_{\mathsf{Att}}(A \cup A', A \wedge A') = (A \cap A'^*) \cup (A' \cap A^*).$$

For the symmetric Conley form we use the following notation: $(A, A') \mapsto A + A'$.

Remark 6.3. The range of the symmetric Conley form is the same as for the standard Conley form. Indeed, if $A' \subset A$ then $\mathsf{C}^*(A, A') = \mathsf{C}_{\mathsf{Att}}(A, A')$. For any pair of attractor A, A' absorption implies that $\mathsf{C}_{\mathsf{Att}}(A, A') = \mathsf{C}_{\mathsf{Att}}(A, A \wedge A')$ which shows that the Conley form can always be determined from nested pairs, in which case the standard and symmetric Conley forms coincide.

6.3. The algebra of attractors

In this section we take a closer look at the algebraic structure of attractors. Algebraic structures and in particular (abelian) group structures are important for the (co)homological theory of sheaves. Our starting point is the lattice of attractors $\mathsf{Att}(\phi)$ of a fixed dynamical system ϕ, which is a bounded, distributive lattice. Before treating the lattice of attractors we first consider bounded, distributive lattices from a more general point of view.

Let $(\mathsf{L}, \wedge, \vee, 0, 1)$ be bounded, distributive lattice. An *ideal* in L is a down-set I such that $a, b \in I$ implies $a \vee b \in I$. If $a \wedge b \in I$ implies $a \in I$ or $b \in I$, then I is called a *prime ideal*. Prime ideals of L are given as $I = f^{-1}(0)$ where $f \in \mathsf{hom}(\mathsf{L}, \mathbf{2})$ and $\mathbf{2}$ is two-elements lattice $\{0, 1\}$, cf. [35,36]. The poset $\big(\Sigma\mathsf{L}, \subset\big)$ of the prime ideals in L, ordered by inclusion, is called the *spectrum* of L. A result due to Birkhoff states that the map $j\colon \mathsf{L} \to \mathsf{O}(\Sigma\mathsf{L})$, given by $a \mapsto j(a) = \{I \in \Sigma\mathsf{L} \mid a \notin I\}$, is a lattice embedding, where $\mathsf{O}(\Sigma\mathsf{L})$ is the lattice of down-sets in $\Sigma\mathsf{L}$ with binary operations $\cap$ and $\cup$, cf. [2,5,6]. In order to characterize the image of j Priestley introduced a topology on $\Sigma\mathsf{L}$ in the spirit of Stone spaces. Consider the basis

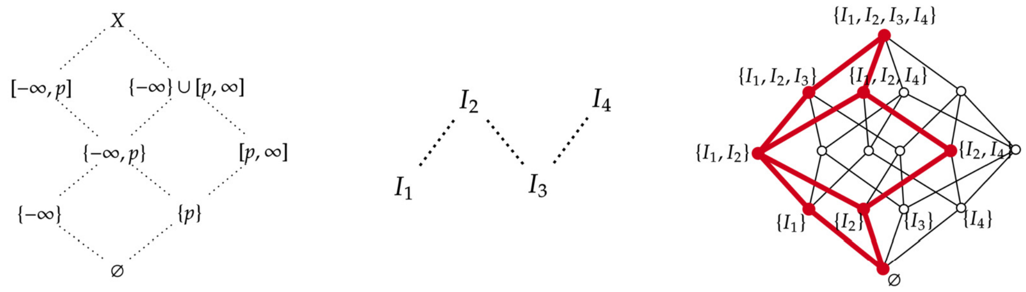


Fig. 3. The saddle-node bifurcation in Fig. 7 has stalk $\mathsf{Att}(\phi)$ represented as Hasse diagram of the lattice [left], cf. [6]. The associated spectrum of prime ideals $I_1 = \{\emptyset, -\infty\}$, $I_2 = \{\emptyset, p\}$, $I_3 = \big\{\emptyset, -\infty, p, \{-\infty, p\}, [-\infty, p]\big\}$ and $I_4 = \big\{\emptyset, p, [p, \infty]\big\}$ [middle]. The (finite) Boolean algebra $\mathsf{BAtt}(\phi)$ [right] is the minimal Boolean extension, or Booleanization of $\mathsf{Att}(\phi)$. In red the isomorphic image of $\mathsf{Att}(\phi)$ in $\mathsf{BAtt}(\phi)$ is indicated. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$\{j(a) \smallsetminus j(b) \mid a, b \in \mathsf{L}\},$$

where $j(a) \smallsetminus j(b) := j(a) \cap j(b)^c$ is set-difference. All the basic open sets are also closed and thus clopen. The Priestley Representation Theorem states that

$$\mathsf{O}^{\mathrm{clp}}(\Sigma\mathsf{L}) = \{j(a) \mid a \in \mathsf{L}\},$$

i.e. L is isomorphic to $\mathsf{O}^{\mathrm{clp}}(\Sigma\mathsf{L})$, the lattice of clopen down-sets, via the map j, cf. [35,36]. The Boolean algebra $\mathsf{BL} := \mathsf{Set}^{\mathrm{clp}}({}^{\circ}\mathsf{L})$ of clopen subsets in $\Sigma\mathsf{L}$ is called the *Booleanization*, or *minimal Boolean extension* of L, and $j\colon \mathsf{L} \to \mathsf{BL}$ is a lattice-embedding given by $j(a) = \{I \in \Sigma\mathsf{L} \mid a \notin I\}$, [35, Thm. 10.19], cf. [37–39]. For more details on Booleanization cf. [5,6].

Boolean algebras can be given the structure of a ring. Given a Boolean algebra $(\mathsf{B}, \wedge, \vee, {}^c, 0, 1)$ define

$$a + b := (a \wedge b^c) \vee (b \wedge a^c) \quad \text{(symmetric difference) and} \quad a \cdot b := a \wedge b.$$

Then, $(\mathsf{B}, +, -, 0, 1)$ is a commutative, idempotent ring (idempotency with respect to multiplication). One retrieves the Boolean algebra structure via $a \vee b = a + b + a \cdot b$. We can formulate this as a faithful functor $\mathsf{I}\colon \mathbf{Bool} \to \mathbf{Ring}$ from the category of Boolean algebras to the category of rings:

$$\mathbf{BDLat} \xrightarrow{\quad \mathsf{B} \quad} \mathbf{Bool} \xrightarrow{\quad \mathsf{I} \quad} \mathbf{Ring}$$

Define the ring obtained from Booleanization of L as the *(Boolean) lattice ring* of L:

$$\mathsf{RL} := (\mathsf{I} \circ \mathsf{B})(\mathsf{L}) \tag{17}$$

the composition is also denoted by $\mathsf{R} := \mathsf{I} \circ \mathsf{B}$. This is the natural way to give an abelian structure to a bounded distributive lattice L. We note that RL is in general not free as additive $\mathbb{Z}_2$-module (vector space), nor as multiplicative monoid. Since L may be regarded as a (commutative) monoid with respect to $\wedge$ we can use the monoid ring construction, cf. [40–42], to define the $\mathbb{Z}_2$-algebra $\mathbb{Z}_2\mathsf{L}$, which is a free $\mathbb{Z}_2$-module (vector space). The elements of $\mathbb{Z}_2\mathsf{L}$ are finite formal sums

$\sum_i a_i$, $a_i \in \mathsf{L}$, with the additional requirement that $2a = a + a = 0$. Multiplication is given by $a \cdot b := a \wedge b$. We refer to $\mathbb{Z}_2\mathsf{L}$ as the *lattice algebra* of L which was first introduced in the context of minimal Boolean extension by MacNeille [43]. The lattice algebra $\mathbb{Z}_2\mathsf{L}$ is clearly a Boolean ring as is the lattice ring. The $\mathbb{Z}_2$-monoid ring construction defines a covariant functor

$$\mathbf{BDLat} \xrightarrow{\ \mathbb{Z}_2\ } \mathbf{Ring}.$$

Example 6.4. Consider the lattice $\mathsf{Att}(\phi)$ shown in Fig. 3. The Boolean algebra of prime ideals $\mathsf{BAtt}(\phi)$ has four atoms, $\{I_1\}, \{I_2\}, \{I_3\}$, and $\{I_4\}$. The corresponding Boolean ring $\mathsf{RAtt}(\phi)$ has sixteen elements, and is generated by these four atoms as a $\mathbb{Z}_2$-vector space $\mathsf{RAtt}(\phi) \cong \mathbb{Z}_2^4$. On the other hand, $\mathbb{Z}_2\mathsf{Att}(\phi)$ is generated by the basis $\{\emptyset, \{-\infty\}, \{p\}, \{-\infty, p\}, [p,\infty], [-\infty, p], \{-\infty\} \cup [p,\infty], X\}$, so $\mathbb{Z}_2\mathsf{Att}(\phi) \cong \mathbb{Z}_2^8$.

The analog of the homomorphism $j : \mathsf{L} \to \mathsf{BL}$ is now given by the ring homomorphism:

$$j : \mathbb{Z}_2\mathsf{L} \to \mathsf{RL}, \qquad j\Big(\sum_i a_i\Big) := \sum_i j(a_i) = \sum_i \alpha_i.$$

By definition $j(a \wedge b) = j(a) \cap j(b)$, which makes j an algebra homomorphism. The image of the generators of $\mathbb{Z}_2\mathsf{L}$ in RL is downsets in $\Sigma\mathsf{L}$ and via the induced $\vee$ operation the lattice L can be retrieved. By construction

$$\begin{aligned} j(a) + j(b) &= \big(j(a) \cup j(b)\big) \smallsetminus \big(j(a) \cap j(b)\big) = j(a \vee b) \smallsetminus j(a \wedge b) \\ &= \mathsf{C}^\sigma(a \vee b, a \wedge b). \end{aligned}$$

When $b \subset a$, then $j(a) + j(b) = \mathsf{C}^\sigma(a, b)$ and thus the sums $j(a) + j(b)$ exhaust the range of the Conley form $\mathsf{C}^\sigma : \mathsf{L} \times \mathsf{L} \to \mathsf{RL}$. We define the set $\mathsf{CL} := \big\{\mathsf{C}^\sigma(a, b) \mid a, b \in \mathsf{L}\big\}$ as the *convexity monoid*: for $\sigma, \sigma' \in \mathsf{CL}$ we have $\sigma \cdot \sigma' = \mathsf{C}^\sigma(a, b) \cap \mathsf{C}^\sigma(a', b') = \mathsf{C}^\sigma(a \wedge a', b \vee b') \in \mathsf{CL}$ and $\sigma \cdot 1 = \mathsf{C}^\sigma(a, b) \cap \mathsf{C}^\sigma(1, 0) = \mathsf{C}^\sigma(a \wedge 1, b \vee 0) = \mathsf{C}^\sigma(a, b) = \sigma$. Clearly the embedding $i : \mathsf{CL} \to \mathsf{RL}$ is a monoid homomorphism.

Lemma 6.5. *The ring homomorphism $j : \mathbb{Z}_2\mathsf{L} \to \mathsf{RL}$ is surjective.*

Proof. Let $\gamma \in \mathsf{RL}$, then by a property of the Priestley topology we can express γ as finite union of the form: $\gamma = \bigcup_i \alpha_i \smallsetminus \alpha_i'$, with $\alpha_i = j(a_i)$, $\alpha_i' = j(a_i')$ and $a_i, a_i' \in \mathsf{L}$. The objective is to prove that γ is in the range of j. Consider $\alpha \smallsetminus \beta \cup \gamma \smallsetminus \delta$. We may assume without loss of generality that $\beta \subset \alpha$ and $\delta \subset \gamma$. Indeed, use $\alpha \smallsetminus \beta = \alpha \smallsetminus (\alpha \cap \beta)$. Therefore, $\alpha \smallsetminus \beta \cup \gamma \smallsetminus \delta = (\alpha + \beta) \cup (\gamma + \delta)$, and

$$\begin{aligned} (\alpha + \beta) \cup (\gamma + \delta) &= \alpha + \beta + \gamma + \delta + (\alpha + \beta) \cap (\gamma + \delta) \\ &= \alpha + \beta + \gamma + \delta + (\alpha \cap \gamma) + (\alpha \cap \delta) + (\beta \cap \gamma) + (\beta \cap \delta), \end{aligned}$$

which corresponds to a sum of j-images of elements in L. We conclude that $\gamma = \bigcup_i \alpha_i \smallsetminus \alpha_i' = \sum_k \tilde{\alpha}_k = \sum_k j(\tilde{a}_k)$, $\tilde{a}_k \in \mathsf{L}$, which proves that γ is in the range of j. □

We now have the following short exact sequence:

$$0 \longrightarrow \ker j \overset{\subset}{\rightarrowtail} \mathbb{Z}_2\mathsf{L} \overset{j}{\twoheadrightarrow} \mathsf{RL} \longrightarrow 0, \tag{18}$$

and since the kernel $\ker j$ is an ideal in $\mathbb{Z}_2\mathsf{L}$ the first isomorphism theorem for rings yields

$$\mathsf{RL} \cong \frac{\mathbb{Z}_2\mathsf{L}}{\ker j},$$

where the isomorphism is given $\sum_i a_i + \ker j \mapsto \sum_i \alpha_i$. If we regard j as a module (vector space) homomorphism then both $\ker j$ and $\mathbb{Z}_2\mathsf{Att}(\phi)$ are free $\mathbb{Z}_2$-modules. The ideal $\ker j$ can be characterized as follows.

Lemma 6.6. $\ker j$ *is the ideal freely generated by elements of the form* $a \vee b + a + b + a \cdot b$.

Proof. For an element $a \vee b + a + b + a \cdot b$ we have that

$$\begin{aligned} j(a \vee b + a + b + a \cdot b) &= j(a \vee b) + j(a) + j(b) + j(a \cdot b) \\ &= j(a) \cup j(b) + j(a) + j(b) + j(a) \cap j(b) \\ &= 2\big(j(a) \cup j(b)\big) = \emptyset, \end{aligned}$$

which proves that finite sums of elements of the form $a \vee b + a + b + a \cdot b$ are contained in $\ker j$. Let $j\left(\sum_i a_i\right) = \sum_i \alpha_i = \emptyset$, then the sum must have an even number of terms. We can rearrange the sequence to a filtration $\alpha'_1 \subset \cdots \subset \alpha'_{2m}$ such that $\sum_i \alpha_i = \sum_i \alpha'_i = \emptyset$. Consequently $\alpha'_{2k-1} + \alpha'_{2k} = \emptyset$ for $k = 1, \cdots, m$, i.e. $\alpha'_{2k-1} = \alpha'_{2k}$ for all k. In order to have distinct elements mapping to $\alpha'_{2j-1} = \alpha'_{2j}$ we have

$$j(b_k + c_k) = \alpha'_{2k-1} = \alpha'_{2k} = j(b_k \vee c_k + b_k \cdot c_k),$$

which proves that element in $\ker j$ is contained in the set of formal sums generated by terms of the form $a \vee b + a + b + a \cdot b$. □

Let us return to the lattice of attractors $\mathsf{Att}(\phi)$. Define the *attractor ring* of a dynamical system ϕ as $\mathsf{RAtt}(\phi) := (\mathsf{I} \circ \mathsf{B})\big(\mathsf{Att}(\phi)\big)$ as the Boolean lattice ring of $\mathsf{Att}(\phi)$. This is the natural way to give an abelian structure to the attractors of a dynamical system. Via the monoid ring construction we obtain the algebra $\mathbb{Z}_2\mathsf{Att}(\phi)$ which is called the *free attractor ring* over $\mathbb{Z}_2$.

7. Sheaf constructions

While sheaves and étalé spaces are equivalent from a categorical viewpoint, the theory of sheaves contributes a rich algebraic toolkit to our study of continuation. Perhaps most prominent is the idea of sheaf cohomology.

7.1. Basic sheaf theory

In this subsection we provide a brief recap of sheaf theory. Let $\mathscr{O}(\mathbf{D})$ be the poset of open subsets in a topological space $\mathbf{D}$ treated as small category where the morphisms are inclusions of sets.

Definition 7.1. A *presheaf* of sets on a topological space $\mathbf{D}$ is a contravariant functor $\mathscr{F} : \mathscr{O}(\mathbf{D}) \to \mathbf{Set}$. Explicitly, a presheaf, $\mathscr{F}$, is characterized by the following ingredients:

(i) for every open subset $\Omega \subset \mathbf{D}$ there is an set $\mathscr{F}(\Omega)$. An element of $\mathscr{F}(\Omega)$ is called a *section* of $\mathscr{F}(\Omega)$ over Ω;
(ii) for every pair of open sets $\Omega' \subseteq \Omega$ in $\mathbf{D}$, there are *restriction morphisms* $\rho_{\Omega',\Omega} : \mathscr{F}(\Omega) \to \mathscr{F}(\Omega')$, which satisfy
  (a) for all open sets Ω in $\mathbf{D}$, it holds that $\rho_{\Omega,\Omega} = \mathrm{id}_{\mathscr{F}(\Omega)}$;
  (b) for all triples of open set $\Omega'' \subseteq \Omega' \subseteq \Omega$ in $\mathbf{D}$ it holds that $\rho_{\Omega'',\Omega} = \rho_{\Omega'',\Omega'} \circ \rho_{\Omega',\Omega}$.

Crucial in the theory of presheaves and sheaves is the notion of stalks.

Definition 7.2. For any $\phi \in \mathbf{D}$ the *stalk* of $\mathscr{F}$ at ϕ is defined as

$$\mathscr{F}_\phi := \varinjlim_{\phi\in\Omega} \mathscr{F}(\Omega).$$

The elements in $\mathscr{F}_\phi$ are called *germs* of sections in $\mathscr{F}(\Omega)$, i.e. $\mathscr{F}(\Omega) \to \mathscr{F}_\phi$.

Sheaves can be defined via presheaves by adding additional axioms on the restriction morphisms with respect to coverings of X.

Definition 7.3. A *sheaf* $\mathscr{F}$ of sets over $\mathbf{D}$ is a presheaf which satisfies the following hypotheses:

(s1) (Mono-presheaf) Let $\{\Omega_i\}_{i\in I}$ be an open covering of an open set Ω in X, and let $\sigma, \sigma' \in \mathscr{F}(\Omega)$ be sections such that

$$\sigma|_{\Omega_i} = \sigma'|_{\Omega_i}, \quad \forall i \in I,$$

then $\sigma = \sigma'$;
(s2) (Gluing) Let $\{\Omega_i\}_{i\in I}$ be an open covering of an open set Ω in X, and let $\{\sigma_i\}_{i\in I}$, $\sigma_i \in \mathscr{F}(\Omega_i)$, be a family of sections such that

$$\sigma_i|_{\Omega_i\cap\Omega_j} = \sigma_j|_{\Omega_i\cap\Omega_j}, \quad \forall i, j \in I,$$

then there exists a section $\sigma \in \mathscr{F}(\Omega)$ with the property that $\sigma_i = \sigma|_{\Omega_i}$.

The section σ given by (s2) is called a *gluing of sections* σ_i, consistent with the overlaps.

There are two important functors between the categories of presheaves and sheaves, denoted by $\mathbf{PrSh}(\mathbf{D})$ and $\mathbf{Sh}(\mathbf{D})$ respectively, which reveal important constructions to turn presheaves into sheaves and vice versa. The construction consists of a number of steps. The first step concerns the definition of étalé spaces, cf. [31, Defn. 3.3].

Definition 7.4. An *étalé space* over a topological space $\mathbf{D}$ is a pair (Π, π), where Π is a topological space and $\pi: \Pi \to \mathbf{D}$ a continuous map such that π is a local homeomorphism. The *set of sections* $\Gamma(\Omega, \Pi)$ consists of continuous maps $\sigma: \Omega \to \Pi$, which satisfy $\pi \circ \sigma = \mathrm{id}_\Omega$.

The morphisms of étalé spaces are maps $f: (\Pi, \pi) \to (\Pi', \pi')$ such that $\pi = \pi' \circ f$. As for presheaves and sheaves, the étalé spaces over $\mathbf{D}$ form a category, denoted by $\mathbf{Et}(\mathbf{D})$. Étalé spaces give rise to sheaves in a natural way. Let (Π, π) be a étalé space, then the presheaf $\Gamma\Pi$ is defined as follows. Let Ω be an open set in $\mathbf{D}$, then

$$\mathscr{O}(\mathbf{D}) \to \mathbf{Set}, \quad \Omega \mapsto (\Gamma\Pi)(\Omega) := \Gamma(\Omega, \Pi), \tag{19}$$

gives a presheaf of sets over $\mathbf{D}$.

Morphisms of étalé spaces (Π, π) yield morphisms of presheaves $\Gamma\Pi$. Let $f: (\Pi, \pi) \to (\Pi', \pi')$ be a étalé morphism, then

$$\Gamma f: \Gamma\Pi \to \Gamma\Pi',$$

defines a presheaf morphism via the relation $\sigma \mapsto f \circ \sigma$.

Proposition 7.5. *Let* (Π, π) *be a étalé space over* $\mathbf{D}$. *Then* $\Gamma\Pi$ *is a sheaf of sets over* $\mathbf{D}$ *and is a called the sheaf of sections of* (Π, π).

This construction defines the functor $\mathsf{\Gamma}: \mathbf{Et}(\mathbf{D}) \to \mathbf{Sh}(\mathbf{D})$. The functor $\mathsf{\Gamma}$ also maps to presheaves. The construction of considering sheaves of sections can be extended to presheaves, which is the second step in establishing the above correspondence.

Let $\mathscr{F}: \mathscr{O}(\mathbf{D}) \to \mathbf{Set}$ be a presheaf of sets over $\mathbf{D}$ and let $\mathscr{F}_\phi$ be the stalk over a point $\phi \in \mathbf{D}$. Define the space

$$\mathrm{L}\mathscr{F} := \bigsqcup_{\phi \in \mathbf{D}} \mathscr{F}_\phi,$$

and $\pi: \mathrm{L}\mathscr{F} \to \mathbf{D}$ the projection such that $\pi^{-1}(\phi) = \mathscr{F}_\phi$. We now put a topology on $\mathrm{L}\mathscr{F}$ such that $(\mathrm{L}\mathscr{F}, \pi)$ is an étalé space. Let Ω be an open set in $\mathbf{D}$ and let $\sigma \in \mathscr{F}(\Omega)$ be a section. Then, define the map $\widehat{\sigma}: \Omega \to \mathrm{L}\mathscr{F}$ via $\phi \mapsto \sigma_\phi \in \mathscr{F}_\phi$. Declare $\widehat{\sigma}(\Omega) = \{\sigma_\phi \mid \phi \in \Omega\}$ as open sets and define

$$\left\{\widehat{\sigma}(\Omega) \mid \sigma \in \mathscr{F}(\Omega)\right\}$$

as a basis for the topology on $\mathrm{L}\mathscr{F}$. With the above defined topology, that $(\mathrm{L}\mathscr{F}, \pi)$ is a étalé space in the sense of Definition 7.4.

Remark 7.6. If we restrict to sheaves and étalé spaces then we have the functors

$$\mathsf{\Gamma}: \mathbf{Et}(\mathbf{D}) \to \mathbf{Sh}(\mathbf{D}), \qquad \mathsf{L}: \mathbf{Sh}(\mathbf{D}) \to \mathbf{Et}(\mathbf{D}).$$

If (Π, π) is a étalé space, then $\mathsf{L}\Gamma\Pi$ and Π are canonically isomorphic as étalé spaces. Similarly, if $\mathscr{F}$ is a sheaf, then $\mathscr{F}$ and $\Gamma\mathsf{L}\mathscr{F}$ are isomorphic. This shows that étalé spaces and sheaves are essentially the same.

Starting from a presheaf $\mathscr{F}$ the consecutive application of Γ and L provides a sheaf $\mathscr{F}^\# := \mathsf{L}\Gamma\mathscr{F}$. This is called the *sheafification* of $\mathscr{F}$ which defines a covariant functor $^\#\colon \mathbf{PrSh}(\mathbf{D}) \to \mathbf{Sh}(\mathbf{D})$. From the previous remark we have that if $\mathscr{F}$ is a sheaf it is canonically isomorphic to its sheafification. For the application of sheaf theory in this text the category $\mathbf{C}$ that is involved often are small categories of algebraic structures such as semi-lattices, lattices, abelian group, rings, etc.

Remark 7.7. Sheaves can be defined with values in categories other than **Set**, such as bounded distributive lattices, rings, or abelian groups. In these cases, achieving the duality between étalé spaces and sheaves requires modification of the definition of étalé space to account for the algebra. For example, we showed in Proposition 6.1 that $\Pi[\mathsf{Att}]$ is a **BDLat**-valued étalé space by demonstrating the wedge and meet operations, as well as the assignments of the neutral elements, were continuous. This ensures that the sheaf associated to the étalé space $\Pi[\mathsf{Att}]$ is in fact a sheaf of bounded distributive lattices. For more details on these kinds of constructions, cf. [44, Sect 1.1], [31, Sect. 2.5].

Example 7.8. Let $\mathsf{E} \in \mathbf{Ab}$ be an abelian group. The presheaf $\mathscr{E}\colon \mathscr{O}(\mathbf{D}) \to \mathbf{Ab}$ defined by $\mathscr{E}(\Omega) := \big\{\sigma\colon \Omega \to \mathsf{E}\ \text{constant}\big\}$, $\Omega \subset \mathbf{D}$ open, is called the *constant presheaf* over $\mathbf{D}$ with values in E. The sheafification $\underline{\mathsf{E}} := \mathscr{E}^\#$ is called the *constant sheaf* over $\mathbf{D}$ with values in E. The constant sheaf can be characterized as the sheaf of locally constant functions with values in E, i.e.

$$\underline{\mathsf{E}}(\Omega) = \big\{\sigma\colon \Omega \to \mathsf{E}\ \text{locally constant}\big\}, \quad \Omega \subset \mathbf{D}, \ \text{open}.$$

If we equip E with the discrete topology then such functions are continuous functions $\sigma\colon \Omega \to \mathsf{E}$. This corresponds to the sheaf of sections of the étalé space $\mathbf{D} \times \mathsf{E}$, with E equipped with the discrete topology, cf. [31, Sect. 2.4], [45, Ex. 3.31 and 3.40]. If $\Omega \subset \mathbf{D}$ is open and connected then $\underline{\mathsf{E}}(\Omega) \cong \mathsf{E}$. For an open set whose connected components are open the $\underline{\mathsf{E}}(\Omega)$ is isomorphic to a direct product of copies of E, one for each connected component, cf. [46, Ex. 1.0.3].

An abelian sheaf $\mathscr{F}$ is called *locally constant* is there exists an open covering $\mathscr{U} = \{\Omega_i\}$ of $\mathbf{D}$ such that $\mathscr{F}|_{\Omega_i}$ is a constant sheaf for all i. This is equivalent to saying that every point allows a neighborhood $\Omega \subset \mathbf{D}$ such that $\mathscr{F}|_\Omega$ is constant, cf. [41, Defn. I.1.9]. Locally constant sheaves are sheaves of sections of covering spaces, [45, Ex. 3.41].

7.2. *Sheafification of continuation*

From an abstract continuation frame we have shown how to build an étalé space which encodes the continuation of the unstable structure of interest. This étalé space $\Pi[\mathsf{G}]$ connects the topology of the base space to the algebraic structure of G. To study this connection, we shift our attention to the sheaves of sections generated by the étalé spaces of continuation frames.

The presheaf

$$\mathscr{S}^{\mathsf{G}}\colon \mathscr{O}(\mathbf{D}) \to \mathbf{Set},$$

where $\mathscr{O}(\mathbf{D})$ is the category of open sets in $\mathbf{D}$, is in fact a sheaf over $\mathbf{D}$ and is called the *sheaf of sections*, cf. [31, Sect. 2.2C]. A stalk of the sheaf $\mathscr{S}^{\mathsf{G}}$ at $\phi \in \mathbf{D}$ is the object $\mathsf{G}(\phi)$. By considering sections in $\Pi[\mathsf{E}]$ we obtain the sheaf of sections $\mathscr{S}^{\mathsf{E}}$ and stalks in $\mathscr{S}^{\mathsf{E}}$ are denoted by $\mathsf{E}(\phi)$, cf. [31, Prop. 3.6].

Remark 7.9. There are multiple equivalent ways to define stalks. The sheaf-theoretic definition is a direct limit $\mathscr{S}^{\mathsf{G}}_\phi := \varinjlim \mathscr{S}^{\mathsf{G}}(\Omega)$ over open neighborhoods Ω of a point ϕ. Equivalently, for an étalé space $\pi\colon \Pi[\mathsf{G}] \to \mathbf{D}$ the stalk at ϕ can be defined as $\pi^{-1}(\phi)$. In our setting, we make the identification between $\pi^{-1}(\phi)$ and $\mathsf{G}(\phi)$.

Lemma 7.10. *Let* $(\mathsf{G}, \mathsf{E}, \mathsf{w})$ *be a continuation frame and let* $\sigma\colon \Omega \to \Pi[\mathsf{G}]$ *be a map with property that* $\pi \circ \sigma = \mathrm{id}$ *on* Ω *(open). Then,* σ *is a section in* $\Pi[\mathsf{G}]$ *if and only if for every* $\phi \in \Omega$ *there exists an open neighborhood* $\Omega_0 \subset \Omega$ *of* ϕ *and* $U \in \mathsf{E}(\phi)$*, such that* $\sigma\big|_{\Omega_0} = \Theta[\mathsf{w}; U]\big|_{\Omega_0}$.

Proof. This follows immediately from the definition of sheaves. □

Sections therefore act locally like $\Theta[\mathsf{w}; U]$. Following this intuition, observe that $\Theta[\mathsf{w}; U]$ is a section in $\Pi[\mathsf{G}]$ over $\Phi[\mathsf{E}; U]$. The above lemma implies we only need to verify that a candidate section locally agrees with $\Theta[\mathsf{w}; U]$ for a *particular* $U \in \mathsf{E}(\phi)$ for some $\phi \in \mathrm{ob}(\mathbf{D})$, rather than all such U. By the same token sections $\sigma\colon \Omega \to \Pi[\mathsf{E}]$ are given locally by $\Theta[\mathrm{id}; U]$, i.e. $\sigma(\phi) = (\phi, U)$.

From the construction of the sheaves $\mathscr{S}^{\mathsf{E}}$ and $\mathscr{S}^{\mathsf{G}}$ we have the following property of the natural transformation w:

$$\begin{array}{ccc} \mathscr{S}^{\mathsf{E}}(\Omega) & \xrightarrow{\mathsf{w}(\Omega)} & \mathscr{S}^{\mathsf{G}}(\Omega) \\ \downarrow{\scriptstyle \rho_{\Omega',\Omega}} & & \downarrow{\scriptstyle \rho_{\Omega',\Omega}} \\ \mathscr{S}^{\mathsf{E}}(\Omega') & \xrightarrow{\mathsf{w}(\Omega')} & \mathscr{S}^{\mathsf{G}}(\Omega') \end{array}$$

where $\mathsf{w}(\Omega)\colon \mathscr{S}^{\mathsf{E}}(\Omega) \to \mathscr{S}^{\mathsf{G}}(\Omega)$ is defined by $\sigma \mapsto \Pi[\mathsf{w}] \circ \sigma$, Ω open, and similarly for $\Omega' \subset \Omega$. The maps $\rho_{\Omega',\Omega}$ are the restriction maps. The latter defines a *morphism of sheaves* $\mathsf{w}\colon \mathscr{S}^{\mathsf{E}} \to \mathscr{S}^{\mathsf{G}}$. Since w yields the stalkwise surjections $\mathsf{w}_\phi\colon \mathsf{E}(\phi) \twoheadrightarrow \mathsf{G}(\phi)$, we say that the morphism $\mathsf{w}\colon \mathscr{S}^{\mathsf{E}} \to \mathscr{S}^{\mathsf{G}}$ is surjective.

7.3. Attractor sheaves

In Section 6.1 we constructed étalé spaces in various categories such as bounded, distributive lattices. The above sheaf of sections construction creates sheaves with values in these same categories. For example, the $\mathbf{C}$-structure $(\mathsf{Att}, \mathsf{ANbhd}, \omega)$ yields the étalé morphism $\Pi[\omega]\colon \Pi[\mathsf{ANbhd}] \twoheadrightarrow \Pi[\mathsf{Att}]$ and the **BDLat**-valued sheaves $\mathscr{S}^{\mathsf{ANbhd}}\colon \mathscr{O}(\mathbf{DS}(\mathbb{T}, \mathbf{D})) \to \mathbf{BDLat}$ and $\mathscr{S}^{\mathsf{Att}}\colon \mathscr{O}(\mathbf{DS}(\mathbb{T}, \mathbf{D})) \to \mathbf{BDLat}$. Hence, we obtain the following morphism of sheaves

$$\omega\colon \mathscr{S}^{\mathsf{ANbhd}} \twoheadrightarrow \mathscr{S}^{\mathsf{Att}}$$

that assigns to every section $\sigma\colon \Omega \to \Pi[\mathsf{ANbhd}]$ the section $\Pi[\omega](\sigma)\colon \Omega \to \Pi[\mathsf{Att}]$. The sheaf $\mathscr{S}^{\mathsf{Att}}$ is called the *attractor lattice sheaf* over $\mathbf{DS}(\mathbb{T}, \mathbf{D})$.

Similarly, we have the morphism of sheaves

$$\alpha\colon \mathscr{S}^{\mathsf{RNbhd}} \twoheadrightarrow \mathscr{S}^{\mathsf{Rep}},$$

where $\mathscr{S}^{\mathsf{Rep}}$ is the *repeller lattice sheaf*. Duality between Att and Rep, as well as between ANbhd and RNbhd, yields the following commutative diagram of sheaves:

$$\begin{array}{ccc} \mathscr{S}^{\mathsf{ANbhd}} & \overset{c}{\longleftrightarrow} & \mathscr{S}^{\mathsf{RNbhd}} \\ {\scriptstyle\omega}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ \mathscr{S}^{\mathsf{Att}} & \overset{*}{\longleftrightarrow} & \mathscr{S}^{\mathsf{Rep}} \end{array}$$

The Conley form on étalé spaces in Section 6.2 gives rise to the **MLat**-valued sheaf

$$\mathscr{S}^{\mathsf{Morse}}\colon \mathscr{O}(\mathbf{DS}(\mathbb{T},\mathbf{D})) \to \mathbf{MLat}.$$

For $\mathscr{S}^{\mathsf{Att}}$, a lattice-valued sheaf, we need a suitable ring structure to define their sheaf cohomology. Viewing $\mathscr{S}^{\mathsf{Att}}$ as a functor from the posetal category of open sets on $\mathbf{DS}(\mathbb{T},X)$ to the category of bounded distributive lattices, we may compose this with a functor from **BDLat** to **Ring**. In Section 6.3 we consider two such functors: the *Boolean ring functor*, $\mathsf{R}=\mathsf{I}\circ\mathsf{B}\colon \mathbf{BDLat}\to\mathbf{Ring}$, and the *monoid ring functor*, $\mathbb{Z}_2\colon \mathbf{Monoid}\to\mathbf{Ring}$. This yields *presheaves*

$$\mathsf{R}\mathscr{S}^{\mathsf{Att}}\colon \mathscr{O}(\mathbf{DS}(\mathbb{T},X)) \to \mathbf{Ring},$$

and

$$\mathbb{Z}_2\mathscr{S}^{\mathsf{Att}}\colon \mathscr{O}(\mathbf{DS}(\mathbb{T},X)) \to \mathbf{Ring}$$

These may fail to be sheaves in general. To remedy this, one studies the sheaf $\left(\mathsf{R}\mathscr{S}^{\mathsf{Att}}\right)^{\#}$ generated by the presheaf $\mathsf{R}\mathscr{S}^{\mathsf{Att}}$ (also called the *sheafification*), cf. Section 7.1. So we define

$$\mathscr{A} := \left(\mathsf{R}\mathscr{S}^{\mathsf{Att}}\right)^{\#}, \qquad \mathscr{A}\!\mathscr{t}\!\mathscr{t} := \left(\mathbb{Z}_2\mathscr{S}^{\mathsf{Att}}\right)^{\#}.$$

These are called the *attractor sheaf* and *free attractor sheaf* over $\mathbf{DS}(\mathbb{T},X)$ respectively. For any ϕ,

$$\mathscr{A}_\phi = \varinjlim_{\Omega\ni\phi}\mathscr{A}(\Omega) \cong \varinjlim_{\Omega\ni\phi}\mathsf{R}\mathscr{S}^{\mathsf{Att}}(\Omega) \cong \mathsf{R}\left(\varinjlim_{\Omega\ni\phi}\mathscr{S}^{\mathsf{Att}}(\Omega)\right) \cong \mathsf{RAtt}(\phi).$$

The first isomorphism is due to sheafification preserving stalks, and the second is due to R preserving colimits (it is a left adjoint). Similarly, $\mathscr{A}\!\mathscr{t}\!\mathscr{t}_\phi \cong \mathbb{Z}_2\mathsf{Att}(\phi)$. Especially for computations in Section 8, it will be useful to describe sections of $\mathscr{A}$ with those of $\mathscr{S}^{\mathsf{Att}}$. After all, on stalks we have the embedding $j\colon \mathsf{Att}(\phi)\to\mathsf{BAtt}(\phi)$ from the attractor lattice to its Booleanization. To that end, we construct a morphism (viewed as between sheaves of sets). Note that as sets, $\mathsf{R}\mathscr{S}^{\mathsf{Att}}(\Omega)$ and $\mathsf{B}\mathscr{S}^{\mathsf{Att}}(\Omega)$ are equal. So for every open set $\Omega\subset\mathbf{DS}(\mathbb{T},X)$, there is a mapping

$$j\colon \mathscr{S}^{\mathsf{Att}}(\Omega)\to \mathsf{R}\mathscr{S}^{\mathsf{Att}}(\Omega),$$

described in 6.3. An application of [6, Proposition 1] ensures that these maps commute with the sheaf restriction maps. Thus, we may collect these maps into a morphism (viewed as between *presheaves* of sets)

$$j\colon \mathscr{S}^{\mathsf{Att}}\to \mathsf{R}\mathscr{S}^{\mathsf{Att}}.$$

Composing this with the natural "sheafification" morphism from $\mathsf{R}\mathscr{S}^{\mathsf{Att}}$ to $(\mathsf{R}\mathscr{S}^{\mathsf{Att}})^{\#}$ yields our desired morphism of sheaves:

$$\ell\colon \mathscr{S}^{\mathsf{Att}}\to \mathscr{A}.$$

Indeed, on stalks, this morphism $\ell_\phi\colon \mathsf{Att}(\phi)\to\mathscr{A}_\phi\cong\mathsf{RAtt}(\phi)$ is precisely the embedding j. An analogous construction may be carried out for the free attractor sheaf, which on stalks may be seen as sending attractors to their corresponding generator in the monoid ring $\mathbb{Z}_2\mathsf{Att}(\phi)$.

Remark 7.11. Similar constructions can be applied to sheaves in other dynamical contexts. The construction via the functor $\mathbb{Z}_2\colon \mathbf{Monoid}\to\mathbf{Ring}$ works for all of the above examples since both bounded, distributive lattices and semilattices compose subcategories of the category of (commutative) monoids. Of particular interest is the *free Morse sheaf*

$$\mathscr{Morse} := \big(\mathbb{Z}_2\mathscr{S}^{\mathsf{Morse}}\big)^{\#}\colon \mathscr{O}(\mathbf{DS}(\mathbb{T},X))\to\mathbf{Ring}.$$

Remark 7.12. The short exact sequence in (18) for Att yields:

$$0 \longrightarrow \ker \jmath \overset{\subset}{\rightarrowtail} \mathscr{Att} \overset{\jmath}{\twoheadrightarrow} \mathscr{A} \longrightarrow 0, \tag{20}$$

where the stalks

$$\mathscr{A}_\phi=\mathsf{RAtt}(\phi)\quad\text{and}\quad \mathscr{Att}_\phi=\mathbb{Z}_2\mathsf{Att}(\phi)$$

are the attractor ring at ϕ and the free attractor ring over $\mathbb{Z}_2$ at ϕ respectively. We define this to be the *fundamental short exact sequence of the attractor sheaf*. The fundamental exact sequence allows us to relate the sheaves $\mathscr{A}$ and $\mathscr{Morse}$. The generators of $\mathscr{Att}$ and the ring structure of $\mathscr{A}$ recover the attractor lattice sheaf $\mathscr{S}^{\mathsf{Att}}$.

Remark 7.13. An alternative way to define the sheaves $\mathscr{A}$ and $\mathscr{Att}$ is a direct definition via étalé spaces. In the case of $\mathscr{Att}$ we define an étalé space $\Pi[\mathbb{Z}_2\mathsf{Att}]$ using the stable $\mathbf{C}$-structure $\big(\mathbb{Z}_2\mathsf{Att},\mathbb{Z}_2\mathsf{ANbhd},\mathbb{Z}_2(\omega)\big)$ via the monoid ring functor. The stability follows from the fact that stability is preserved under free sums. We obtain the étalé space $\pi:\Pi[\mathbb{Z}_2\mathsf{Att}]\to\mathbf{DS}(\mathbb{T},X)$ and the associated sheaf of sections $\mathscr{S}^{\mathbb{Z}_2\mathsf{Att}}$. It holds that $\mathscr{S}^{\mathbb{Z}_2\mathsf{Att}}\cong\mathscr{Att}$. For the Boolean ring functor it is more involved to prove that $\big(\mathsf{RAtt},\mathsf{RANbhd},\mathsf{R}(\omega)\big)$ is a continuation frame, but $\mathscr{S}^{\mathsf{RAtt}}\cong\mathscr{A}$.

7.4. Finite sublattice and Morse representation sheaves

Following Lemma 5.8, we also have an **Lat**-valued étalé space $\Pi[\mathrm{sub_F}\mathsf{Att}]$, encoding the continuation of finite sublattices of attractors. As earlier, we can consider the corresponding sheaf of sections $\mathscr{S}^{\mathrm{sub_F}\mathsf{Att}}\colon \mathbf{DS}(\mathbb{T}, X) \to \mathbf{Lat}$. For an open set $\Omega \subset \mathbf{DS}(\mathbb{T}, X)$, a section in $\mathscr{S}^{\mathrm{sub_F}\mathsf{Att}}(\Omega)$ assigns to each dynamical system $\phi \in \Omega$ a finite sublattice of $\mathsf{A} \subset \mathsf{Att}(\phi)$. The lattice operations on $\mathscr{S}^{\mathrm{sub_F}\mathsf{Att}}(\Omega)$, on stalks, send two finite sublattices to their intersection or the smallest sublattice containing both. This yields the following question concerning the structure of the sheaf $\mathscr{S}^{\mathrm{sub_F}\mathsf{Att}}$:

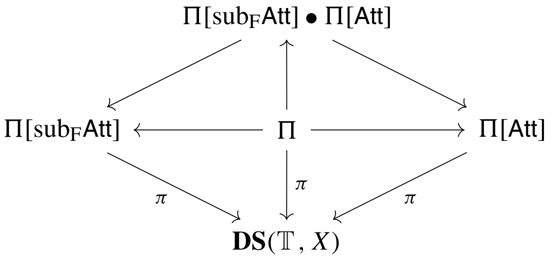
Can we view sections of $\mathscr{S}^{\mathrm{sub_F}\mathsf{Att}}$ as a lattice of sections of $\mathscr{S}^{\mathsf{Att}}$?

This is not always possible, see Example 9.6.

To understand this structure one needs to be able to relate the sheaves $\mathscr{S}^{\mathrm{sub_F}\mathsf{Att}}$ and $\mathscr{S}^{\mathsf{Att}}$. Define the following étalé space on $\mathbf{DS}(\mathbb{T}, X)$:

$$\Pi := \big\{(\phi, \mathsf{A}, A) \in \Pi[\mathrm{sub_F}\mathsf{Att}] \bullet \Pi[\mathsf{Att}] : A \in \mathsf{A}\big\}.$$

The projection from $\Pi[\mathrm{sub_F}\mathsf{Att}] \bullet \Pi[\mathsf{Att}]$ remains a surjective local homeomorphism when restricted to the subspace Π. There is a commutative diagram of restriction maps for étalé spaces:



Denote the sheaf of sections associated to the étalé space Π by

$$\mathscr{E}^{\mathrm{sub_F}\mathsf{Att}}\colon \mathscr{O}(\mathbf{DS}(\mathbb{T}, X)) \to \mathbf{Set}.$$

A section of $\mathscr{E}^{\mathrm{sub_F}\mathsf{Att}}$ traces out the continuation of a finite sublattice of attractors, as well as a specific attractor in that sublattice. From the diagram, there are two morphisms:

$$q\colon \mathscr{E}^{\mathrm{sub_F}\mathsf{Att}} \to \mathscr{S}^{\mathsf{Att}} \qquad r\colon \mathscr{E}^{\mathrm{sub_F}\mathsf{Att}} \to \mathscr{S}^{\mathrm{sub_F}\mathsf{Att}},$$

which restricts sections to their attractor and finite sublattice components respectively. For an open set $\Omega \subset \mathbf{DS}(\mathbb{T}, X)$ and a section $\nu \in \mathscr{S}^{\mathrm{sub_F}\mathsf{Att}}(\Omega)$, we can consider the set $r_\Omega^{-1}(\nu)$ consisting of sections of $\mathscr{E}^{\mathrm{sub_F}\mathsf{Att}}$ which agree with ν on their finite sublattice component. This set has a bounded distributive lattice structure, defined on the attractor component. Suppose $\sigma, \sigma' \in r_\Omega^{-1}(\nu)$:

$$\sigma(\phi) = (\nu(\phi), A), \quad \sigma'(\phi) = (\nu(\phi), A'), \qquad (\sigma \wedge \sigma')(\phi) := (\nu(\phi), A \wedge A').$$

The meet operation is defined similarly. We can then pass this lattice through q, and achieve a bounded distributive lattice of sections of $\mathscr{S}^{\mathsf{Att}}$:

$$q_\Omega\colon r_\Omega^{-1}(\nu) \subset \mathscr{E}^{\mathrm{sub_F}\mathsf{Att}}(\Omega) \to \mathscr{S}^{\mathsf{Att}}(\Omega)$$

We make the following observations:

- For any section $\sigma \in q_\Omega\big(r_\Omega^{-1}(\nu)\big)$, with $\sigma(\phi) = (\phi, A)$, we have that $A \in \mathsf{A}$, where $\nu(\phi) = (\phi, \mathsf{A})$. In other words, the value of σ at ϕ is contained in the value of ν at ϕ.
- Composing the stalk restriction map $\rho_\phi\colon \mathscr{S}^{\mathsf{Att}}(\Omega) \to \mathsf{Att}(\phi)$ yields the composite lattice homomorphism

$$f_{\Omega,\phi}\colon q_\Omega\big(r_\Omega^{-1}(\nu)\big) \to \mathsf{A} \subset \mathsf{Att}(\phi),$$

  where $\nu(\phi) = (\phi, \mathsf{A})$.
- If $f_{\Omega,\phi}$ is surjective at every $\phi \in \Omega$, we retrieve ν from these sections:

$$\nu(\phi) = \big(\phi, \{A_\sigma\}\big), \quad \text{where} \quad \sigma \in q_\Omega\big(r_\Omega^{-1}(\nu)\big), \quad \sigma(\phi) = (\phi, A_\sigma).$$

Proposition 7.14. *Let $\nu \in \mathscr{S}^{\mathrm{sub_F}\mathsf{Att}}(\Omega)$ for some open set $\Omega \subset \mathbf{DS}(\mathbb{T}, X)$ and let $\phi \in \Omega$. Then, there is an open neighborhood Ω' of ϕ such that $f_{\Omega',\phi'}$ defined by $\nu\big|_{\Omega'}$ is surjective for all $\phi' \in \Omega'$.*

Proof. By Lemma 7.10 ν yields a neighborhood Ω' of ϕ upon which $\nu\big|_{\Omega'} = \Theta[\boldsymbol{\omega}; \mathsf{N}]\big|_{\Omega'}$ for some $\mathsf{N} \in \mathrm{sub_F}\mathsf{ANbhd}(\phi)$. For each $U \in \mathsf{N}$, we have a section $\Theta[\omega; U]\big|_{\Omega'} \in \mathscr{S}^{\mathsf{Att}}(\Omega')$. Indeed, we can define the following section in $\mathscr{E}^{\mathrm{sub_F}\mathsf{Att}}(\Omega')$:

$$\phi \mapsto \big(\phi, \boldsymbol{\omega}_\phi(\mathsf{N}), \omega_\phi(U)\big)$$

which maps to $\Theta[\omega; U]\big|_{\Omega'}$ under q, and therefore $\Theta[\omega; U]\big|_{\Omega'} \in q_{\Omega'}\big(r_{\Omega'}^{-1}(\nu)\big)$. Let $\phi' \in \Omega'$, and $A \in \mathsf{A}$, where $\nu(\phi') = (\phi', \mathsf{A})$. Then $A = \omega_\psi(U)$ for some $U \in \mathsf{N}$, since $\boldsymbol{\omega}_\psi(\mathsf{N}) = \mathsf{A}$. Moreover, since $f_{\Omega',\phi'}\big(\Theta[\omega; U]\big|_{\Omega'}\big) = A$ for arbitrary choices of ϕ' and A, the proof is complete. □

Proposition 7.14 justifies that *locally* a section in $\mathscr{S}^{\mathrm{sub_F}\mathsf{Att}}$ may be interpreted as a finite distributive lattice of sections in $\mathscr{S}^{\mathsf{Att}}$. We will investigate when this interpretation extends globally at a later stage.

Dually, we can consider the continuation frame $\big(\mathsf{MRepr}, \mathsf{MTess}, \boldsymbol{\Delta}\big)$, which defines the **Lat**-valued *Morse representation sheaf* $\mathscr{S}^{\mathsf{MRepr}}$, cf. Sect. 5.2, which generalizes the construction in [23]. Applying Proposition 4.9 to the natural transformation $\boldsymbol{\mu}\colon \mathrm{sub_F}\mathsf{Att} \to \mathsf{MRepr}$ with stable extension $\boldsymbol{\tau}\colon \mathrm{sub_F}\mathsf{ANbhd} \to \mathsf{MTess}$ yields a sheaf isomorphism

$$\boldsymbol{\mu}\colon \mathscr{S}^{\mathrm{sub_F}\mathsf{Att}} \to \mathscr{S}^{\mathsf{MRepr}}.$$

The lattice structure of $\mathscr{S}^{\mathsf{MRepr}}$ allows common coarsings and refinements of Morse representations: let $\sigma_{\mathsf{M}}, \sigma'_{\mathsf{M}} \in \mathscr{S}^{\mathsf{MRepr}}(\Omega)$, then

$$\sigma_{\mathsf{M}} \vee \sigma'_{\mathsf{M}} \in \mathscr{S}^{\mathsf{MRepr}}(\Omega) \quad \text{and} \quad \sigma_{\mathsf{M}} \wedge \sigma'_{\mathsf{M}} \in \mathscr{S}^{\mathsf{MRepr}}(\Omega),$$

the *common coarsening* and *common refinement* of Morse representations continuations respectively. The binary operations are defined in the sheaf $\mathscr{S}^{\mathrm{sub_F}\mathsf{Att}}$ via

$$\boldsymbol{\mu}^{-1}(\sigma_{\mathsf{M}}) \wedge \boldsymbol{\mu}^{-1}(\sigma'_{\mathsf{M}}) \quad \text{and} \quad \boldsymbol{\mu}^{-1}(\sigma_{\mathsf{M}}) \vee \boldsymbol{\mu}^{-1}(\sigma'_{\mathsf{M}}),$$

respectively. We can dualize the earlier theory for $\mathscr{S}^{\mathrm{sub_F}\mathsf{Att}}$ to describe sections of $\mathscr{S}^{\mathsf{MRepr}}$. For a section $\zeta \in \mathsf{MRepr}(\Omega)$, there is a corresponding section $\nu := \boldsymbol{\mu}(\zeta) \in \mathscr{S}^{\mathrm{sub_F}\mathsf{Att}}(\Omega)$. We again have $f_{\Omega,\phi}\colon q_\Omega(r_\Omega^{-1}(\nu)) \to \mathsf{A}$ where $\nu(\phi) = (\phi, \mathsf{A})$. Suppose $q_\Omega(r_\Omega^{-1}(\nu))$ is finite. We can dualize to achieve:

$$g_{\Omega,\phi}\colon \mathsf{M}(\mathsf{A}) \to \mathsf{P}_\Omega,$$

where P_Ω denotes the poset of join-irreducible elements of $q_\Omega(r_\Omega^{-1}(\nu))$. The map $g_{\Omega,\phi}$ composes the isomorphism between $\mathsf{J}(\mathsf{A})$, the join-irreducible elements of A, and $\mathsf{M}(\mathsf{A})$ with the dual of $f_{\Omega,\phi}$. The Morse representation $\mathsf{M}(\mathsf{A})$ is exactly the value of ζ at ϕ, in other words, $\zeta(\phi) = (\phi, \mathsf{M}(\mathsf{A}))$. If the lattice morphism $f_{\Omega,\phi}$ is surjective, then $g_{\Omega,\phi}$ is an embedding and thus a *Morse decomposition*, cf. [6, Def. 7]. Thus we get an analogous statement to Proposition 7.14.

Corollary 7.15. *Let $\zeta \in \mathscr{S}^{\mathsf{MRepr}}(\Omega)$ for an open $\Omega \subset \mathbf{DS}(\mathbb{T}, X)$ such that $q_\Omega(r_\Omega^{-1}(\boldsymbol{\mu}_\Omega(\zeta)))$ is finite, and $\phi \in \Omega$. Then there is an open neighborhood Ω' of ϕ such that $g_{\Omega',\phi'}$ is a Morse decomposition for all $\phi' \in \Omega'$.*

8. Parameter spaces and pullbacks

In this section we discuss continuation frames for parametrized families of dynamical systems and how the associated sheaves can be constructed.

8.1. Parametrized dynamical systems

Let Λ be a topological space. In keeping with the spirit of the paper we keep the conditions mild but in practical situations Λ is a CW-space.

Definition 8.1. Let X be a compact topological space. A *parametrized dynamical system over* Λ on X is a continuous map $\boldsymbol{\phi}\colon \mathbb{T} \times X \times \Lambda \to X$ such that $\phi^\lambda := \boldsymbol{\phi}(\cdot,\cdot,\lambda) \in \mathbf{DS}(\mathbb{T}, X)$ for all $\lambda \in \Lambda$.

The category of dynamical systems $\mathbf{DS}(\mathbb{T}, X)$ is a function space equipped with the compact-open topology. For a parametrized dynamical system $\boldsymbol{\phi}$ we define the *transpose* $\boldsymbol{\phi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X)$ by

$$\boldsymbol{\phi}_*(\lambda) = \phi^\lambda := \boldsymbol{\phi}(\cdot,\cdot,\lambda).$$

The transpose $\boldsymbol{\phi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X)$ is a continuous map without additional assumptions on the topological spaces Λ and X, cf. D.

For the continuation frame $(\mathsf{Att}, \mathsf{ANbhd}, \omega)$ on $\mathbf{DS}(\mathbb{T}, X)$ a parametrized dynamical system yields a pullback étalé space on Λ:

$$\boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Att}] := \left\{(\lambda, \phi, A) \in \Lambda \times \Pi[\mathsf{Att}] \mid \boldsymbol{\phi}_*(\lambda) = \pi(\phi, A) = \phi\right\},$$

i.e. the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Att}] & \xrightarrow{(\lambda,\phi,A)\mapsto(\phi,A)} & \Pi[\mathsf{Att}] \\ \Big\downarrow{\scriptstyle (\lambda,\phi,A)\mapsto\lambda} & & \Big\downarrow{\scriptstyle \pi} \\ \Lambda & \xrightarrow{\boldsymbol{\phi}_*} & \mathbf{DS}(\mathbb{T}, X) \end{array}$$

where $\boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Att}]$ is the pullback in the category of topological spaces, cf. [44, Sect. I.3]. From [32, Prop. 2.4.9] it follow that $\boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Att}] \to \Lambda$ is an étalé space over Λ. The binary operations on $\Pi[\mathsf{Att}]$ can be verified to be continuous on the inverse image étalé space. As before we obtain the following **BDLat**-valued pullback sheaf

$$\boldsymbol{\phi}_*^{-1}\mathscr{S}^{\mathsf{Att}} \colon \mathscr{O}(\Lambda) \to \mathbf{BDLat},$$

as the sheaf of sections of $\boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Att}]$. Applying the Boolean ring functor R to the sheaf of sections yields a ring valued sheaf:

$$\mathscr{A}^{\boldsymbol{\phi}_*} := (\mathsf{R}\boldsymbol{\phi}_*^{-1}\mathscr{S}^{\mathsf{Att}})^{\#} \colon \mathscr{O}(\Lambda) \to \mathbf{Ring}.$$

The *ringed space* $(\Lambda, \mathscr{A}^{\boldsymbol{\phi}_*})$ encodes the continuation data of attractors for the parametrized dynamical system. Similarly, for the monoid ring functor $\mathbb{Z}_2\cdot$ we obtain:

$$\mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\phi}_*} := (\mathbb{Z}_2\boldsymbol{\phi}_*^{-1}\mathscr{S}^{\mathsf{Att}})^{\#} \colon \mathscr{O}(\Lambda) \to \mathbf{Ring},$$

where the multiplication is inherited from the monoidal structure of Att, cf. Section 6.3. We are now in the setting of sheaf cohomology. Since the category of sheaves of abelian groups has enough injectives, the *ith sheaf cohomology groups* may be defined as the right derived functors of the global section functor. A more direct and detailed construction can be found in [44]. We apply these, and their relative versions, to the sheaves $\mathscr{A}^{\boldsymbol{\phi}_*}$ and $\mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\phi}_*}$. Theorem 8.7 and the later sections will show the cohomology groups $H^i(\Lambda, \mathscr{A}^{\boldsymbol{\phi}_*})$ and $H^i(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\phi}_*})$ are algebraic invariants which can detect bifurcations.

Remark 8.2. The sheaf $\mathscr{A}^{\boldsymbol{\phi}_*}$ can be alternatively defined as follows. The Boolean ring functor yields the étalé space $\Pi[\mathsf{RAtt}]$ and the associated sheaf of sections $\boldsymbol{\phi}_*^{-1}\mathscr{S}^{\mathsf{RAtt}}$, which defines the sheaf as the pullback sheaf with respect to $\boldsymbol{\phi}_*$.

8.2. Conjugate dynamical systems and homeomorphic étalé spaces

We start off with the basic notion of conjugacy in dynamical systems.

Definition 8.3. Let X and Y be compact topological spaces, and let $\boldsymbol{\phi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X)$ and $\boldsymbol{\psi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, Y)$ be parametrized dynamical systems. A *conjugacy* between $\boldsymbol{\phi}_*$ and $\boldsymbol{\psi}_*$ is a continuous map $h\colon \Lambda \times X \to Y$ and a continuous reparametrization $\tau\colon \Lambda \times \mathbb{T} \times X \to \mathbb{T}$, such that

(i) $h^\lambda \times \tau^\lambda := h(\lambda, \cdot) \times \tau(\lambda, \cdot, \cdot)$ is a conjugacy in $\mathsf{hom}(\phi^\lambda, \psi^\lambda)$ for all $\lambda \in \Lambda$;
(ii) $h^\lambda(X_i) = Y_i$ uniformly for all $\lambda \in \Lambda$, where X_i and Y_i are the connected components of X and Y respectively.

If a conjugacy h exists, then $\boldsymbol{\phi}_*$ and $\boldsymbol{\psi}_*$ are said to be *conjugate parametrized dynamical systems*.

Remark 8.4. Assumption (ii) is always satisfied pointwise for λ by appropriately indexing the components of X and Y. The uniformity in the above definition is not guaranteed since no restrictions on the topology of Λ are required. For specific topologies on Λ condition (ii) may be superfluous.

Remark 8.5. One may also consider quasiconjugacies between parametrized dynamical systems over Λ.

Since h^λ is a conjugacy we know from Remark 3.5 that the push-forward $U^\lambda \mapsto h^\lambda(U^\lambda)$ is an attracting neighborhood for ψ_λ and similarly, the push-forward $A^\lambda \mapsto h^\lambda(A^\lambda)$ is an attractor for ψ_λ.

Lemma 8.6. *The following diagram commutes:*

$$\begin{array}{ccc} \mathsf{ANbhd}(\phi^\lambda) & \overset{\cong}{\longleftrightarrow} & \mathsf{ANbhd}(\psi^\lambda) \\ {\scriptstyle \omega_{\phi^\lambda}}\big\downarrow & & \big\downarrow{\scriptstyle \omega_{\psi^\lambda}} \\ \mathsf{Att}(\phi^\lambda) & \overset{\cong}{\longleftrightarrow} & \mathsf{Att}(\psi^\lambda) \end{array}$$

Proof. Indeed, the maps from above we have $U^\lambda \mapsto h^\lambda(U^\lambda) \mapsto \omega_{\psi^\lambda}\big(h^\lambda(U^\lambda)\big)$. From below yields $U^\lambda \mapsto A^\lambda = \omega_{\phi^\lambda}(U^\lambda) \mapsto h^\lambda(A^\lambda)$. Since h is a conjugacy it follows from Remark 3.5 that Lemma 3.2 applies to both h^λ and $(h^\lambda)^{-1}$. This gives:

$$\omega_{\psi^\lambda}\big(h^\lambda(U^\lambda)\big) = \omega_{\psi^\lambda}\Big(h^\lambda\big(\omega_{\phi^\lambda}(U^\lambda)\big)\Big) = \omega_{\psi^\lambda}\big(h^\lambda(A^\lambda)\big) = h^\lambda(A^\lambda), \tag{21}$$

which proves commutativity. □

Lemma 8.6 holds for all $\lambda \in \Lambda$ and which provides stalkwise isomorphisms between the associated sheaves of attractors. This however does not give isomorphic sheaves necessarily!

Theorem 8.7 *(Conjugacy Invariance Theorem). Let X, Y be compact metric spaces. Suppose $\boldsymbol{\phi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X)$ and $\boldsymbol{\psi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, Y)$ are conjugate parametrized dynamical systems. Then, the étalé spaces $\boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Att}]$ and $\boldsymbol{\psi}_*^{-1}\Pi[\mathsf{Att}]$ are homeomorphic.*

Proof. From Lemma 8.6 we have the following commutative diagram of maps:

$$\begin{array}{ccc} \boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Att}] & \xrightarrow{h_*} & \boldsymbol{\psi}_*^{-1}\Pi[\mathsf{Att}] \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \Lambda & \end{array}$$

where h_* is defined by $(\lambda, \phi^\lambda, A^\lambda) \mapsto h_*(\lambda, \phi^\lambda, A^\lambda) := \big(\lambda, \psi^\lambda, h^\lambda(A^\lambda)\big)$. It is sufficient to show continuity, since if h_* is continuous, then h_* is a local homeomorphism, in which case h_*^{-1} is a also a local homeomorphism (h_* is a bijection), cf. [32, Prop. 2.4.8]. This proves that $\boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Att}]$ and $\boldsymbol{\psi}_*^{-1}\Pi[\mathsf{Att}]$ are homeomorphic.

In order to prove continuity we argue as follows. Consider the following commutative diagram

$$\begin{array}{ccc} & & \boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Att}] \\ & {\scriptstyle \boldsymbol{\phi}_*^{-1}\Theta[\omega;U]}\nearrow & \downarrow{\scriptstyle\pi} \\ \boldsymbol{\phi}_*^{-1}\Phi[\mathsf{ANbhd};U] & \xrightarrow{\subset} & \Lambda \end{array}$$

where $\boldsymbol{\phi}_*^{-1}\Phi[\mathsf{ANbhd};U] = \big\{\lambda \mid U \in \mathsf{ANbhd}(\phi^\lambda)\big\}$ and $\boldsymbol{\phi}_*^{-1}\Theta[\omega;U](\lambda) = \big(\lambda, \phi^\lambda, \omega_{\phi^\lambda}(U)\big)$. Let $D_0 \subset \Lambda$ be an open neighborhood of $\lambda_0 \in \Lambda$ and let

$$\boldsymbol{\psi}_*^{-1}\Theta\big[\omega; h^{\lambda_0}(U^{\lambda_0})\big](D_0) = \Big\{\Big(\lambda, \psi^\lambda, \omega_{\psi^\lambda}\big(h^{\lambda_0}(U^{\lambda_0})\big)\Big) \mid \lambda \in D_0\Big\}$$

be an open neighborhood of $h_*\big(\lambda_0, \phi^\lambda, A^{\lambda_0}\big) = \big(\lambda_0, \psi^\lambda, h^{\lambda_0}(A^{\lambda_0})\big)$ in $\boldsymbol{\psi}_*^{-1}\Pi[\mathsf{Att}]$ for some compact $U^{\lambda_0} \in \mathsf{ANbhd}(\phi^{\lambda_0})$. In order to establish continuity we seek a neighborhood $D_0' \subset D_0 \subset \Lambda$ such that

$$\begin{aligned} h_*\Big(\boldsymbol{\phi}_*^{-1}\Theta[\omega; U^{\lambda_0}](D_0')\Big) &= \Big\{\Big(\lambda, \psi^\lambda, h^\lambda\big(\omega_{\phi^\lambda}(U^{\lambda_0})\big)\Big) \mid \lambda \in D_0'\Big\} \\ &= \Big\{\Big(\lambda, \psi^\lambda, \omega_{\psi^\lambda}\big(h^\lambda(U^{\lambda_0})\big)\Big) \mid \lambda \in D_0'\Big\} \subset \boldsymbol{\psi}_*^{-1}\Theta\big[\omega; h^{\lambda_0}(U^{\lambda_0})\big](D_0), \end{aligned}$$

where the second equality follows from Lemma 8.6, Eqn. (21). This is equivalent to saying

$$\omega_{\psi^\lambda}\big(h^\lambda(U^{\lambda_0})\big) = \omega_{\psi^\lambda}\big(h^{\lambda_0}(U^{\lambda_0})\big), \quad \forall \lambda \in D_0'.$$

For notational convenience we write

$$U := h^{\lambda_0}(U^{\lambda_0}) \in \mathsf{ANbhd}(\psi^{\lambda_0}), \quad \text{and} \quad A = h^{\lambda_0}(A^{\lambda_0}) = \omega_{\psi^{\lambda_0}}(U') \in \mathsf{Att}(\psi^{\lambda_0}).$$

We rephrase the above condition as:

$$\omega_{\psi^\lambda}\big(h^\lambda(U^{\lambda_0})\big) = \omega_{\psi^\lambda}(U), \quad \forall \lambda \in D_0'. \tag{22}$$

For $U^{\lambda_0} = \emptyset$, or for $U^{\lambda_0} = \bigsqcup_i X_i \subset X$, any union of connected components of X, Eqn. (22) is satisfied by the uniform conjugacy condition in Defn. 8.3(ii), cf. Remark 8.4. For the remainder of the proof we assume $U^{\lambda_0} \neq \emptyset$ and $U^{\lambda_0} \neq \bigsqcup_i X_i$, for all unions of connected components of X. Therefore, we may carry out the arguments for the components $U_i^{\lambda_0} = U^{\lambda_0} \cap X_i \neq \emptyset, X_i$.

Choose a compact attracting neighborhood $U' \in \mathsf{ANbhd}(\psi^{\lambda_0})$ such that $U' \subset \operatorname{int} U$ and $\omega_{\psi^{\lambda_0}}(U') = A$. Indeed, since A is an attractor $\operatorname{cl} U^c \cap A = \emptyset$, cf. [2, Lemma 3.23]. Therefore there exist open sets N, N' such that $A \subset N$, $\operatorname{cl} U^c \subset N'$ and $N \cap N' = \emptyset$. As a matter of fact $\operatorname{cl} N \cap N' = \emptyset$. Define $U' = \operatorname{cl} N$. By construction $A^* \subset U^c \subset \operatorname{cl} U^c \subset N'$ and thus $U' \cap A^* = \emptyset$ which proves that (i) $\omega_{\psi^{\lambda_0}}(U') = A$, (ii) $A \subset N \subset U'$, (iii) $U' = \operatorname{cl} N \subset N'^c \subset \left(\operatorname{cl} U^c\right)^c = \operatorname{int} U$, and thus U' is an attracting neighborhood satisfying the properties stated above, cf. [2, Lemma 3.21]. From the fact that $U \neq \bigsqcup_i Y_i$, a union of components, it follows that $\operatorname{int} U \subsetneq U$. Thus by Property (iii) there exists a $\delta_1 > 0$ such that $B_{\delta_1}(U') \subset U$ and therefore $d_{\mathsf{H}}(U, U') \geqslant \delta_1 > 0$, where d_{H} is the Hausdorff metric on the space $\mathsf{H}(X)$ of compact subsets of X.

By the same token we can choose a compact repelling neighborhood $V \in \mathsf{RNbhd}(\psi^{\lambda_0})$ such that $V \cap U = \emptyset$ and $\omega_{\psi^{\lambda_0}}(V^c) = A$. Indeed, repeat the above arguments starting with $U \cap A^* = \emptyset$. V is compact, so there exists a $\delta_2 > 0$ such that $d_{\mathsf{H}}(U, V) \geqslant \delta_2 > 0$.

Since, $\boldsymbol{\psi}_*^{-1}\Theta\left[\omega; U\right]$, $\boldsymbol{\psi}_*^{-1}\Theta\left[\omega; U'\right]$ and $\boldsymbol{\psi}_*^{-1}\Theta\left[\omega; V^c\right]$ define local sections in $\boldsymbol{\psi}_*^{-1}\Pi[\mathsf{Att}]$ over $\boldsymbol{\psi}_*^{-1}\Phi[\mathsf{ANbhd}; U]$, $\boldsymbol{\psi}_*^{-1}\Phi[\mathsf{ANbhd}; U']$ and $\boldsymbol{\psi}_*^{-1}\Phi[\mathsf{ANbhd}; V^c]$ respectively, and since

$$\boldsymbol{\psi}_*^{-1}\Theta\left[\omega; U\right](\lambda_0) = \boldsymbol{\psi}_*^{-1}\Theta\left[\omega; U'\right](\lambda_0) = \boldsymbol{\psi}_*^{-1}\Theta\left[\omega; V^c\right](\lambda_0)$$

there exists an open set $E_0 \subset \Lambda$ on which three sections coincide, i.e.

$$B^\lambda := \omega_{\psi^\lambda}(U) = \omega_{\psi^\lambda}(U') = \omega_{\psi^\lambda}(V^c), \quad \forall \lambda \in E_0,$$

and $B^\lambda \subset \operatorname{int} U$, $B^\lambda \subset \operatorname{int} U'$ and $B^\lambda \subset \operatorname{int} V^c$ for all $\lambda \in E_0$.

Let $\widetilde{U}$ be any compact neighborhood such that $d_{\mathsf{H}}(U, \widetilde{U}) < \delta = \min\{\delta_1, \delta_2\}/2$ and let $\lambda \in E_0$. Then,

$$B^\lambda \subset U' \subset \widetilde{U}, \qquad \widetilde{U} \cap (B^\lambda)^* \subset \widetilde{U} \cap V = \emptyset,$$

which by [2, Lemma 3.21] implies that $\omega_{\psi^\lambda}(\widetilde{U}) = B^\lambda$ for all $\lambda \in E_0$.

Finally, using the continuity of h_{H}^λ in Lemma D.1, choose an open sets $D_0' \subset E_0 \cap D_0$ such that $d_{\mathsf{H}}\left(h^\lambda(U^{\lambda_0}), U\right) < \delta$ for all $\lambda \in D_0'$. By the previous we choose $\widetilde{U} = h^\lambda(U^{\lambda_0})$ which proves that

$$\omega_{\psi^\lambda}\left(h^\lambda(U^{\lambda_0})\right) = B^\lambda = \omega_{\psi^\lambda}(U), \quad \forall \lambda \in D_0',$$

establishing (22) and thereby the theorem. □

Remark 8.8. The condition that the spaces X and Y are compact metric spaces is used at several places in the proof and in particular for using the Hausdorff metric. The characterizations of attracting and repelling neighborhoods via attractors and dual repellers at least work in compact Hausdorff spaces.

Theorem 8.7 can be extended to other structures. Since $\boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Att}]$ is homeomorphic (as a sheaf of sets) to $\boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Rep}]$, we can get a homeomorphism between $\boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Rep}]$ and $\boldsymbol{\psi}_*^{-1}\Pi[\mathsf{Rep}]$. There is the following commutative diagram for Morse sets:

$$\begin{array}{ccc}
\boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Att}] \bullet \boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Att}] & \longrightarrow & \boldsymbol{\psi}_*^{-1}\Pi[\mathsf{Att}] \bullet \boldsymbol{\psi}_*^{-1}\Pi[\mathsf{Att}] \\
\Big\downarrow \Pi[\mathsf{C}_{\mathsf{Att}}] & & \Big\downarrow \Pi[\mathsf{C}_{\mathsf{Att}}] \\
\boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Morse}] & \longrightarrow & \boldsymbol{\psi}_*^{-1}\Pi[\mathsf{Morse}] \\
& \searrow \quad \swarrow & \\
& \Lambda &
\end{array}$$

where the top horizontal map is given by

$$(\lambda, \phi^\lambda, A), (\lambda, \phi^\lambda, A') \mapsto (\lambda, \psi^\lambda, h^\lambda(A)), (\lambda, \psi^\lambda, h^\lambda(A'))$$

and the bottom horizontal map is given by

$$(\lambda, \phi^\lambda, M) \mapsto (\lambda, \psi^\lambda, h^\lambda(M)),$$

which, using a similar argument to Proposition 4.9, establishes that the étalé spaces $\boldsymbol{\phi}_*^{-1}\Pi[\mathsf{Morse}]$ and $\boldsymbol{\psi}_*^{-1}\Pi[\mathsf{Morse}]$ are homeomorphic.

Corollary 8.9. *Let X and Y be homeomorphic compact metric spaces and let Att_X and Att_Y be the attractor functors on $\mathbf{DS}(\mathbb{T}, X)$ and $\mathbf{DS}(\mathbb{T}, Y)$ respectively. Then, the étalé spaces $\Pi[\mathsf{Att}_X]$ and $\Pi[\mathsf{Att}_Y]$ are homeomorphic.*

Proof. Let $h\colon X \to Y$ be a homeomorphism and let $\Lambda = \mathbf{DS}(\mathbb{T}, X)$. Then, $\boldsymbol{\phi}_*$ is the identity map. The map $\boldsymbol{\psi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, Y)$ is defined as follows: $\Lambda \ni \phi \mapsto h \circ \phi \circ h^{-1} = \psi$. Then,

$$h\big(\phi_t(x)\big) = h\Big(\phi_t\big(h^{-1}(y)\big)\Big) = \psi_t(y) = \psi_t\big(h(x)\big),$$

which proves that $\boldsymbol{\phi}_*$ and $\boldsymbol{\psi}_*$ are conjugate parametrized dynamical systems. □

9. Bifurcations and sheaf cohomology

Sheaves attach both local and global data to a topological space. In our setting of continuation, they encode how dynamical structures vary with parameter values on open sets. Oftentimes, given an open cover of the topological space, one can glue together the local information on each element of the cover to obtain global information.

However, sometimes local information fails to extend globally. Sheaf cohomology, which can be viewed as a generalization of singular cohomology, is a powerful tool for studying this. An interpretation for singular cohomology groups is that they constitute obstructions to a topological space being contractible. Sheaf cohomology generalizes this by representing barriers for local sections to extend to global sections.

One can always solve an attractor's continuation *locally* using an attracting neighborhood. But this problem is sometimes impossible *globally*. Sheaf cohomology provides a framework for quantifying when and how this occurs. Together with the conjugacy invariance theorem, this will build an algebraic invariant for parametrized dynamical systems, which can be used to study bifurcations.

Recall that a parametrized dynamical system on a topological space Λ is a continuous map $\boldsymbol{\phi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X)$ such that $\boldsymbol{\phi}_*(\lambda)\colon \mathbb{T} \times X \to X$ is a dynamical system for all $\lambda \in \Lambda$. In principle Λ may be $\mathbf{DS}(\mathbb{T}, X)$ but in practice simpler topological spaces for Λ are used. In this section, to utilize Theorem 8.7, *we assume X is a compact metric space.*

Definition 9.1. A parametrized dynamical system $\boldsymbol{\phi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X)$ is *stable* at a point $\lambda_0 \in \Lambda$ if there exists an open neighborhood $\Lambda' \ni \lambda_0$ such that $\boldsymbol{\phi}_*\big|_{\Lambda'}$ is conjugate to the constant parametrization $\boldsymbol{\theta}_*\colon \Lambda' \to \mathbf{DS}(\mathbb{T}, X)$, given by $\lambda \mapsto \boldsymbol{\phi}_*(\lambda_0)$ for all $\lambda \in \Lambda'$. If λ_0 is not stable, it is called a *bifurcation point*. A parametrized dynamical system $\boldsymbol{\phi}_*$ is *stable* on a subset $\Lambda' \subset \Lambda$ if it is stable at every point in $\Lambda' \subset \Lambda$.

If a parametrized dynamical system $\boldsymbol{\phi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X)$ is conjugate to the constant parametrization $\boldsymbol{\theta}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X)$ on Λ it is called *uniformly stable*.

In general stability of a parametrized dynamical system does not imply uniform stability. For instance if Λ is not connected then $\boldsymbol{\phi}_*$ need not be conjugate to a fixed constant system $\boldsymbol{\theta}_*$. This example indicates that stability does not imply uniform stability in general if Λ is disconnected. See Example 9.5 for an counter example with a connected space Λ.

9.1. Locally constant sheaves

Let $\boldsymbol{\phi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X)$ be a parametrized dynamical system. From the previous we have the induced attractor sheaf and free attractor sheaf over Λ:

$$\mathscr{A}^{\boldsymbol{\phi}_*}\colon \mathscr{O}(\Lambda) \to \mathbf{Ring}, \qquad \mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\phi}_*}\colon \mathscr{O}(\Lambda) \to \mathbf{Ring}.$$

The ringed spaces $(\Lambda, \mathscr{A}^{\boldsymbol{\phi}_*})$ and $(\Lambda, \mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\phi}_*})$ encode the continuation data of attractors for the parametrized dynamical system. At a later stage we also include the attracting neighborhood sheaf and free attracting neighborhood sheaf $\mathscr{N}$ and $\mathscr{ANbhd}$ respectively.

Recall that for an abelian group $\mathsf{E} \in \mathbf{Ab}$ the presheaf $\mathscr{E}\colon \mathscr{O}(\Lambda) \to \mathbf{Ab}$ defined by $\mathscr{E}(\Lambda') := \big\{\sigma\colon \Lambda' \to \mathsf{E}\ \text{constant}\big\}$, $\Lambda' \subset \Lambda$ open, is called the *constant presheaf* over Λ with values in E. The sheafification $\underline{\mathsf{E}} := \mathscr{E}^\#$ is called the *constant sheaf* over Λ with values in E. The constant sheaf can be characterized as the sheaf of locally constant functions with values in E, i.e.

$$\underline{\mathsf{E}}(\Lambda') = \big\{\sigma\colon \Lambda' \to \mathsf{E}\ \text{locally constant}\big\}, \quad \Lambda' \subset \Lambda,\ \text{open}.$$

If we equip E with the discrete topology then such functions are continuous functions $\sigma\colon \Lambda' \to \mathsf{E}$. This corresponds to the sheaf of section of the étalé space $\Lambda \times \mathsf{E}$, with E equipped with the discrete topology, cf. [31, Sect. 2.4], [45, Ex. 3.31 and 3.40]. If $\Lambda' \subset \Lambda$ is open and connected then $\underline{\mathsf{E}}(\Lambda') \cong \mathsf{E}$. For an open set whose connected components are open then $\underline{\mathsf{E}}(\Lambda')$ is isomorphic to a direct product of copies of E, one for each connected component, cf. [46, Ex. 1.0.3]. An abelian sheaf $\mathscr{F}$ is called *locally constant* is there exists an open covering $\{\Lambda_i\}$ of Λ such that $\mathscr{F}|_{\Lambda_i}$ is a constant sheaf for all i. This is equivalent to saying that every point allows a neighborhood $\Lambda' \subset \Lambda$ such that $\mathscr{F}|_{\Lambda'}$ is constant, cf. [41, Defn. I.1.9]. Locally constant sheaves are sheaves of sections of covering spaces, [45, Ex. 3.41].

Lemma 9.2. *Let $\boldsymbol{\theta_*}\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X)$ be a constant parametrization. Then, the sheaves $\mathscr{A}^{\boldsymbol{\theta_*}}$ and $\mathscr{Att}^{\boldsymbol{\theta_*}}$ are constant sheaves.*

Proof. The pullback étalé space $\boldsymbol{\theta}_*^{-1}\Pi[\mathsf{Att}]$ is given by

$$\boldsymbol{\theta}_*^{-1}\Pi[\mathsf{Att}] \cong \Lambda \times \mathsf{A},$$

where $\mathsf{A} = \mathsf{Att}(\phi^{\lambda_0})$, for some $\lambda_0 \in \Lambda$, is given the discrete topology. Therefore the sheaf of sections $\boldsymbol{\theta}_*^{-1}\mathscr{S}^{\mathsf{Att}}$ is a constant sheaf. Consequently, $\mathscr{A}^{\boldsymbol{\theta_*}}$ and $\mathscr{Att}^{\boldsymbol{\theta_*}}$ are also constant sheaves. □

Lemma 9.3. *Let $\boldsymbol{\phi_*}\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X)$ be stable. Then, the sheaves $\mathscr{A}^{\boldsymbol{\phi_*}}$ and $\mathscr{Att}^{\boldsymbol{\phi_*}}$ are locally constant sheaves.*

Proof. Pick a point $\lambda_0 \in \Lambda$. Since $\boldsymbol{\phi_*}$ is stable there exists a neighborhood $\Lambda' \ni \lambda_0$ such that $\boldsymbol{\phi_*}|_{\Lambda'}$ is conjugate to the constant parametrization. By the Conjugacy Invariance Theorem in 8.7 we have that $\mathscr{A}^{\boldsymbol{\phi_*}}|_{\Lambda'} \cong \mathscr{A}^{\boldsymbol{\theta_*}}|_{\Lambda'}$ as sheaves. The latter is a constant sheaf over Λ' and therefore $\mathscr{A}^{\boldsymbol{\phi_*}}|_{\Lambda'}$ is a constant sheaf over Λ' by definition. We conclude that $\mathscr{A}^{\boldsymbol{\phi_*}}$ is locally constant. The same applies to $\mathscr{Att}^{\boldsymbol{\phi_*}}$. □

Remark 9.4. If $\boldsymbol{\phi_*}$ is uniformly stable then $\boldsymbol{\phi_*}$ is conjugate to a constant parametrization $\boldsymbol{\theta_*}$ on Λ. The associated étalé spaces are homeomorphic by Theorem 8.7 and thus the sheaves $\mathscr{A}^{\boldsymbol{\phi_*}}$ and $\mathscr{Att}^{\boldsymbol{\phi_*}}$ are constant sheaves is this case.

Example 9.5. Let X be the 2-point compactification of the line and consider the following family of differential equations

$$\dot{x} = \sin(x+\lambda), \quad x \in \mathbb{R}, \;\; \lambda \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

The above system defines a 1-parameter family of flows $\boldsymbol{\phi_*}\colon \Lambda \to \mathbf{DS}(\mathbb{R}, X)$ of flows on X over parameter space $\Lambda = \mathbb{S}^1$. Via the conjugacy $x \mapsto x - \lambda$ we conclude that $\boldsymbol{\phi_*}$ is stable and thus the attractor sheaf $\mathscr{A}^{\boldsymbol{\phi_*}}$ is a locally constant sheaf as indicated by Lemma 9.3. Since $\pm\infty$ are not attractors, the only global sections in $\mathscr{A}^{\boldsymbol{\phi_*}}$ are $\emptyset$ and X. The stalks of $\mathscr{A}^{\boldsymbol{\phi_*}}$ are infinite complete, atomic Boolean algebras which proves that $\mathscr{A}^{\boldsymbol{\phi_*}}$ is not a constant sheaf.

The above example shows that even if Λ is connected, then a stable system need not be uniformly stable. Indeed, $\boldsymbol{\phi_*}$ in Example 9.5 allows a conjugacy over $\Lambda = \mathbb{S}^1$, then the attractor sheaf $\mathscr{A}^{\boldsymbol{\phi_*}}$ is constant which contradicts above statement that $\mathscr{A}^{\boldsymbol{\phi_*}}$ is locally constant but not constant.

Example 9.6. Define a vector field restricted to the compact subset $[-2,2] \times [-2,2]$ of $\mathbb{R}^2$:

$$F(x,y) = (-x(x+1)(x-1), -y).$$

We can rotate the vector field with a parameter θ:

$$F_\theta(x,y) = R_{-\theta} F(R_\theta(x,y)),$$

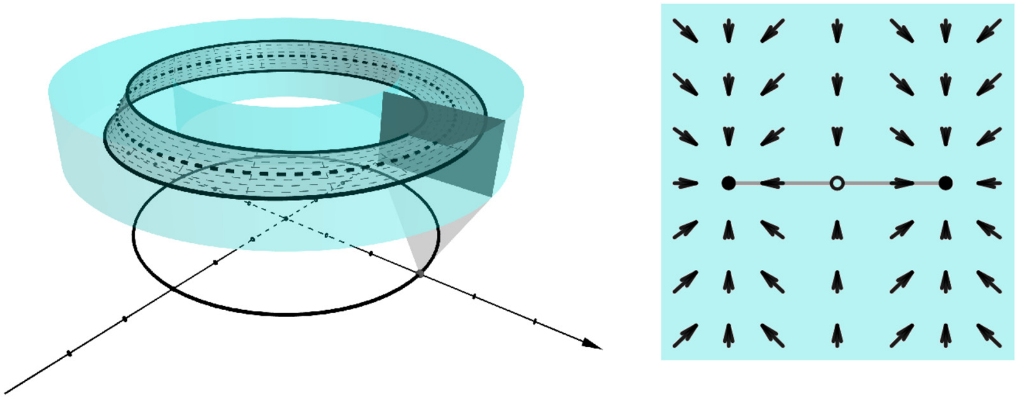


Fig. 4. An illustration of Example 9.6. The vector field at the cross section is rotated over the parameter space $\mathbb{S}^1$. The system is stable, so the attractor sheaf is locally constant. However, it is impossible to continue either attracting fixed point globally.

where R_θ denotes the rotation matrix of angle θ. Since $F_\pi(x, y) = F_0(x, y)$, gluing at 0 and π (not 2π) gives us a parametrized dynamical system $\boldsymbol{\phi_*}\colon \mathbb{S}^1 \to \mathbf{DS}(\mathbb{R}, [0, 2] \times [0, 2])$ by integrating the vector field. The invariant set $[-1, 1] \times \{0\}$ undergoes a half-twist over $\mathbb{S}^1$. There are only three global sections of $\boldsymbol{\phi_*^{-1}}\mathscr{S}^{\mathsf{Att}}$:

$$\theta \mapsto \emptyset, \quad \theta \mapsto R_\theta(\{(1, 0), (-1, 0)\}), \quad \theta \mapsto R_\theta([-1, 1] \times \{0\}).$$

Alas, each stalk is a five element lattice and $\mathbb{S}^1$ is connected, so $\boldsymbol{\phi_*^{-1}}\mathscr{S}^{\mathsf{Att}}$ is not the constant sheaf. Additionally, the five element attractor lattice is a global section in $\boldsymbol{\phi_*^{-1}}\mathscr{S}^{\mathsf{sub_F Att}}$, but cannot be represented as a collection of global sections of $\boldsymbol{\phi_*^{-1}}\mathscr{S}^{\mathsf{Att}}$. (Fig. 4.)

As pointed out above, a locally constant sheaf is the sheaf of sections of a covering space. With additional conditions on Λ such sheaves may be constant sheaves.

Proposition 9.7 *(cf. [47], Prop. 4.20 and [45], Prop. 7.5). Let Λ be a simply connected and locally path connected topological space, and let $\mathscr{F}$ be a locally constant sheaf of rings on Λ. Then, $\mathscr{F}$ is a constant sheaf.*

The same statement holds for contractible spaces Λ, cf. [48, Exer. II.4]. We can apply the above proposition to the attractor sheaf $\mathscr{A}^{\boldsymbol{\phi_*}}$ and free attractor sheaf $\mathscr{Att}^{\boldsymbol{\phi_*}}$ for simple parametrized systems $\boldsymbol{\phi_*}$.

Corollary 9.8. *Let $\boldsymbol{\phi_*}\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X)$ be stable and let Λ be a simply connected and locally path connected topological space. Then, $\mathscr{A}^{\boldsymbol{\phi_*}}$ and $\mathscr{Att}^{\boldsymbol{\phi_*}}$ are constant sheaves.*

For constant sheaves the sheaf cohomology can be related to singular cohomology which is a useful tool in our treatment of bifurcations.

Proposition 9.9 *(cf. [49], Thm. 9). Let Λ be a locally contractible topological space ([50, p. 57]), and let R be an arbitrary ring. If $\underline{\mathsf{R}}$ denotes the constant sheaf with values in R, then $H^k(\Lambda; \underline{\mathsf{R}}) \cong H^k_{\mathrm{sing}}(\Lambda; \mathsf{R})$ for all k.*

If we combine Lemma 9.3, Corollary 9.8 and Proposition 9.9 we obtain a result that determines the sheaf cohomology of the attractor sheaves for simple parametrized dynamical systems.

Corollary 9.10. *Let $\boldsymbol{\phi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X)$ be stable and let Λ be a locally contractible and simply connected topological space. Then,*

$$H^k(\Lambda; \mathscr{A}^{\boldsymbol{\phi}_*}) \cong H^k_{\text{sing}}(\Lambda; \mathscr{A}^{\boldsymbol{\phi}_*}_{\lambda_0}), \quad \forall k,$$

*where $\mathscr{A}^{\boldsymbol{\phi}_*}_{\lambda_0} \in$ **Ring** is a stalk at any $\lambda_0 \in \Lambda$. A similar statement holds for $H^k(\Lambda; \mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\phi}_*})$.*

Proof. Lemma 9.3 implies that $\mathscr{A}^{\boldsymbol{\phi}_*}$ is a constant sheaf. A locally contractible space is locally simply connected and locally path connected, but not necessarily simply connected. In combination with the condition of simple connectedness we can combine Corollary 9.8 and Proposition 9.9, which completes the proof. □

9.2. Sufficient conditions

The statements about sheaf cohomology in Section 9.1 imply the following sufficient condition for bifurcations to exist. The theorems stated for the attractor sheaf $\mathscr{A}^{\boldsymbol{\phi}_*}$ can also be stated for the free attractor sheaf $\mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\phi}_*}$.

Theorem 9.11. *Let Λ be both contractible and locally contractible. Suppose that*

$$H^k(\Lambda; \mathscr{A}^{\boldsymbol{\phi}_*}) \neq 0, \quad \textit{for some} \quad k > 0.$$

Then, there exists a bifurcation point in $\lambda_0 \in \Lambda$.

Proof. Suppose there are no bifurcation points. This implies that $\boldsymbol{\phi}_*$ is stable which by Corollary 9.10 implies that $H^k(\Lambda; \mathscr{A}^{\boldsymbol{\phi}_*}) \cong H^k_{\text{sing}}(\Lambda; \mathsf{R})$ for all k (where R is isomorphic to a stalk of $\mathscr{A}^{\boldsymbol{\phi}_*}$). Since Λ is contractible, we have that $H^k_{\text{sing}}(\Lambda; \mathsf{R}) = 0$ for all $k > 0$. Combining these statements yields that $H^k(\Lambda; \mathscr{A}^{\boldsymbol{\phi}_*}) \cong H^k_{\text{sing}}(\Lambda; \mathsf{R}) = 0$ for all $k > 0$, which contradicts the above assumptions. □

As we will see in Section 10 the above criterion does not always detect bifurcations. In order to get a more in depth look into local bifurcations we consider its relative sheaf cohomology for $\mathscr{A}^{\boldsymbol{\phi}_*}$. We use the following lemma about long exact sequences in sheaf cohomology.

Lemma 9.12. *Let $\mathscr{F}$ be a sheaf of rings on Λ and let $\Lambda' \overset{i}{\hookrightarrow} \Lambda$. Assume that the induced homomorphisms $i^k_*\colon H^k(\Lambda; \mathscr{F}) \to H^k(\Lambda'; \mathscr{F})$ are isomorphisms for all $k \geqslant 0$. Then,*

$$H^k(\Lambda, \Lambda'; \mathscr{F}) \cong 0, \quad \forall k \geqslant 0.$$

Proof. For triple $(\Lambda', \emptyset) \overset{i}{\hookrightarrow} (\Lambda, \emptyset) \overset{j}{\hookrightarrow} (\Lambda, \Lambda')$ we have the long exact sequence,

$$0 \xrightarrow{\delta^0} H^0(\Lambda, \Lambda'; \mathscr{F}) \xrightarrow{j_*^0} H^0(\Lambda; \mathscr{F}) \xrightarrow{i_*^0} H^0(\Lambda'; \mathscr{F}) \xrightarrow{\delta^1} H^1(\Lambda, \Lambda'; \mathscr{F}) \xrightarrow{j_*^1} H^1(\Lambda; \mathscr{F}) \xrightarrow{i_*^1} H^1(\Lambda'; \mathscr{F}) \xrightarrow{\delta^2} \cdots .$$

For the exactness of the maps and the isomorphisms i_*^k we have: $\ker j_*^0 = \operatorname{im} \delta^0 = 0$, which proves that j_*^0 is injective. Furthermore, since i_*^0 is an isomorphism we have $\ker i_*^0 = 0 = \operatorname{im} j_*^0$ and thus $H^0(\Lambda, \Lambda'; \mathscr{F}) \cong 0$. The remaining relative homology groups are determined as follows: $\ker \delta^1 = \operatorname{im} i_*^0 = H^0(\Lambda'; \mathscr{F}) \cong H^0(\Lambda; \mathscr{F})$. Therefore, $\ker j_*^1 = \operatorname{im} \delta^1 = 0$, which shows that j_*^1 is injective. Furthermore, $\ker i_*^1 = 0 = \operatorname{im} j_*^1$, consequently $H^1(\Lambda, \Lambda'; \mathscr{F}) \cong 0$. The same argument can be repeated now for all other k. □

As an immediate consequence of the long exact sequence we have the following corollary if we apply Lemma 9.12 to the attractor sheaf $\mathscr{A}^{\boldsymbol{\phi}_*}$.

Corollary 9.13. *Suppose $H^k(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\phi}_*}) \neq 0$ for some k. Then, there exist $k_0 \geqslant 0$ for which the inclusion i does not imply an isomorphism $i_*^{k_0} : H^{k_0}(\Lambda; \mathscr{A}^{\boldsymbol{\phi}_*}) \to H^{k_0}(\Lambda'; \mathscr{A}^{\boldsymbol{\phi}_*})$.*

The relative sheaf cohomology can be used to formulate an analogous criterion as Theorem 9.11.

Theorem 9.14. *Let Λ be both contractible and locally contractible, and let $\Lambda' \subset \Lambda$ be a deformation retract of Λ with $\boldsymbol{\phi}_*$ stable on Λ'. Suppose that*

$$H^k(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\phi}_*}) \neq 0, \quad \text{for some} \quad k \geqslant 0.$$

Then, there exists a bifurcation point in $\lambda_0 \in \Lambda \smallsetminus \Lambda'$.

Proof. Suppose there are no bifurcation points in $\Lambda \smallsetminus \Lambda'$. This implies that $\boldsymbol{\phi}_*$ is stable on Λ. Since Λ is contractible and locally contractible, it is simply connected and locally path connected. It follows from Proposition 9.8 that $\mathscr{A}^{\boldsymbol{\phi}_*}$ is a constant sheaf on Λ. Since Λ' is a deformation retract of Λ, the same holds for Λ' and $\mathscr{A}^{\boldsymbol{\phi}_*}|_{\Lambda'} \cong \mathscr{A}^{\boldsymbol{\phi}_*}$. This implies that $H^0(\Lambda; \mathscr{A}^{\boldsymbol{\phi}_*}) \cong H^0(\Lambda'; \mathscr{A}^{\boldsymbol{\phi}_*})$. By Corollary 9.10, since $i_*^k : H^k(\Lambda; \mathsf{R}) \to H^k(\Lambda'; \mathsf{R})$ is an isomorphism for all k, we have that $H^k(\Lambda; \mathscr{A}^{\boldsymbol{\phi}_*}) \cong H^k(\Lambda'; \mathscr{A}^{\boldsymbol{\phi}_*}) \cong 0$ for all $k \geqslant 1$. Combining these statements gives $H^k(\Lambda; \mathscr{A}^{\boldsymbol{\phi}_*}) \cong H^k(\Lambda'; \mathscr{A}^{\boldsymbol{\phi}_*})$ for all k. This implies by Lemma 9.12 that $H^k(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\phi}_*}) \cong 0$ for all k, which contradicts the assumption that $H^k(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\phi}_*}) \neq 0$ for some k. Therefore, $\boldsymbol{\phi}_*$ is not stable on $\Lambda \smallsetminus \Lambda'$ and there exists a bifurcation point $\lambda_0 \in \Lambda \smallsetminus \Lambda'$. □

In the interest of working with computations, we introduce some terminology for the attractor sheaf $\mathscr{A}^{\boldsymbol{\phi}_*}$. Recall the morphism $\ell : \mathscr{S}^{\mathsf{Att}} \to \mathscr{A}$, which on stalks is exactly the embedding $j : \mathsf{Att}(\phi) \to \mathsf{RAtt}(\phi)$. After pulling back by a parametrization $\boldsymbol{\phi}_*$, we get a morphism

$$\ell : \boldsymbol{\phi}_*^{-1} \mathscr{S}^{\mathsf{Att}} \to \mathscr{A}^{\boldsymbol{\phi}_*}.$$

Fix an open set Ω. Then the following is a collection of sections in $\mathscr{A}^{\boldsymbol{\phi}_*}(\Omega)$:

$$\mathscr{B}_\Omega := \{\ell(\sigma) : \sigma \text{ is a join-irreducible element of } \boldsymbol{\phi}_*^{-1}\mathscr{S}^{\mathsf{Att}}(\Omega)\}.$$

When $\mathscr{B}_\Omega$ is a basis for $\mathscr{A}^{\boldsymbol{\phi}_*}(\Omega)$ as a vector space over $\mathbb{Z}_2$, we refer to this as the *canonical basis*. This happens often in our computations when the lattices of sections are finite, and when the sheafification morphisms are isomorphisms on Ω.

10. Examples of one-parameter bifurcations

In this section we discuss a number of standard one-parameter bifurcations such as a saddle-node bifurcation and a pitchfork bifurcation. We will also examine bifurcation at multiple bifurcation points. The objective is to show that sheaf cohomology picks up bifurcations. At a later stage we will discuss the more practical side of computing sheaf cohomology from limited data.

10.1. One-parameter bifurcations at a single parameter value

In this subsection we list three fundamental bifurcations in one-parameter systems. We apply the above results to compute the sheaf cohomology and to compare the criteria. For example if $\Lambda = \mathbb{R}$ or $\Lambda = I$, a bounded interval, then the above theorem applies. This is of interest for one-parameter bifurcations. The following lemma addresses the case where $\boldsymbol{\phi}_*$ has one bifurcation point on $\mathbb{R}$, which will assist in computations.

Lemma 10.1. *Let $\mathscr{F}$ be a sheaf of rings on $\Lambda = \mathbb{R}$, such that $\mathscr{F}$ is a constant sheaf on both $(-\infty, \lambda_0)$ and (λ_0, ∞) for some $\lambda_0 \in \mathbb{R}$. Then, $\mathscr{F}$ is acyclic, i.e. $H^k(\Lambda, \mathscr{F}) = 0$ for all $k \geqslant 1$, and $\Gamma(\mathscr{F}) \cong \mathscr{F}_{\lambda_0}$.*

Proof. Let $\epsilon > 0$ and let B_ϵ denote the interval $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. There is a restriction cohomomorphism $r\colon \mathscr{F} \rightsquigarrow \mathscr{F}\big|_{B_\epsilon}$. We will show this induces an isomorphism of cohomology:

$$r^*\colon H^*(\mathbb{R}; \mathscr{F}) \to H^*\big(B_\epsilon; \mathscr{F}\big|_{B_\epsilon}\big). \tag{23}$$

First we address global sections. Because $\mathscr{F}$ is constant on $(-\infty, \lambda_0)$ and (λ_0, ∞), sections in $\Gamma(\mathscr{F}\big|_{B_\epsilon})$ extend uniquely to sections in $\Gamma(\mathscr{F})$. Thus, $r_0^*\colon \Gamma(\mathscr{F}) \to \Gamma(\mathscr{F}\big|_{B_\epsilon})$ is an isomorphism. For $k > 1$, $H^k(\mathbb{R}; \mathscr{F})$ and $H^k\big(B_\epsilon; \mathscr{F}\big|_{B_\epsilon}\big)$ vanish, since intervals have covering dimension 1, cf. [48, Lemma 2.7.3 and Proposition 3.2.2]. So the maps

$$r_k^*\colon H^k(\mathbb{R}; \mathscr{F}) \to H^k\big(B_\epsilon; \mathscr{F}\big|_{B_\epsilon}\big)$$

are trivially isomorphisms. Now we consider $k = 1$. Let $\mathbb{R}^* = \mathbb{R} \setminus \{\lambda_0\}$, so that B_ϵ and $\mathbb{R}^*$ form a cover of $\mathbb{R}$. Note that $\mathscr{F}\big|_{\mathbb{R}^*}$ is locally constant, with vanishing higher cohomology groups. There is a Mayer-Vietoris exact sequence:

$$0 \longrightarrow \Gamma(\mathscr{F}) \xrightarrow{\alpha} \Gamma\big(\mathscr{F}\big|_{B_\epsilon}\big) \oplus \Gamma\big(\mathscr{F}\big|_{\mathbb{R}^*}\big) \xrightarrow{\beta} \Gamma\big(\mathscr{F}\big|_{B_\epsilon \cap \mathbb{R}^*}\big) \xrightarrow{\delta} H^1(\mathbb{R};\mathscr{F}) \longrightarrow H^1\big(B_\epsilon;\mathscr{F}\big|_{B_\epsilon}\big) \oplus H^1\big(\mathbb{R}^*;\mathscr{F}\big|_{\mathbb{R}^*}\big) \longrightarrow H^1\big(B_\epsilon \cap \mathbb{R}^*;\mathscr{F}\big|_{B_\epsilon \cap \mathbb{R}^*}\big) \longrightarrow 0.$$

Since $H^1(\mathbb{R};\mathscr{F}\big|_{\mathbb{R}^*})$ and $H^1(B_\epsilon \cap \mathbb{R}^*;\mathscr{F}\big|_{B_\epsilon \cap \mathbb{R}^*})$ vanish the sequence simplifies to:

$$0 \longrightarrow \Gamma(\mathscr{F}) \xrightarrow{\alpha} \Gamma\big(\mathscr{F}\big|_{B_\epsilon}\big) \oplus \Gamma\big(\mathscr{F}\big|_{\mathbb{R}^*}\big) \xrightarrow{\beta} \Gamma\big(\mathscr{F}\big|_{B_\epsilon \cap \mathbb{R}^*}\big) \xrightarrow{\delta} H^1(\mathbb{R};\mathscr{F}) \xrightarrow{r_1^*} H^1(B_\epsilon;\mathscr{F}\big|_{B_\epsilon}) \longrightarrow 0.$$

The map β is surjective, since the restriction from $\Gamma\big(\mathscr{F}\big|_{\mathbb{R}^*}\big)$ to $\Gamma\big(\mathscr{F}\big|_{B_\epsilon \cap \mathbb{R}^*}\big)$ is surjective. Following the sequence yields $\operatorname{Im}\delta = \ker r_1^* = 0$. $\operatorname{im} r_1^* = \ker 0 = H^1(B_\epsilon;\mathscr{F}\big|_{B_\epsilon})$, so r_1^* is also surjective. This implies that the restriction cohomomorphism $r\colon \mathscr{F} \rightsquigarrow \mathscr{F}\big|_{B_\epsilon}$ induces an isomorphism on cohomology and establishes (23). Indeed, for $\epsilon' < \epsilon$, the restriction cohomomorphism from $\mathscr{F}\big|_{B_\epsilon}$ to $\mathscr{F}\big|_{B_{\epsilon'}}$ is an isomorphism, again giving an isomorphism of cohomology. So,

$$H^*\big(\mathbb{R};\mathscr{F}\big) \approx \varinjlim_{\epsilon>0} H^*\big(B_\epsilon;\mathscr{F}\big|_{B_\epsilon}\big).$$

We can compute the limit using [44, Theorem II.10.6]:

$$\varinjlim_{\epsilon>0} H^*\big(B_\epsilon;\mathscr{F}\big|_{B_\epsilon}\big) \approx H^*\big(\{\lambda_0\};\mathscr{F}\big|_{\{\lambda_0\}}\big).$$

Since $\mathscr{F}\big|_{\{\lambda_0\}}$ is flasque (restriction maps are surjective), it is acyclic. This also implies $\Gamma(\mathscr{F}) \cong \Gamma\big(\mathscr{F}\big|_{\{\lambda_0\}}\big) \cong \mathscr{F}_{\lambda_0}$, completing the proof. □

The same results hold for $\Lambda = I$, a bounded, or semi-bounded interval.

Lemma 10.2. *Let $\mathscr{F}$ be a sheaf of rings on Λ and let $\Lambda' \overset{i}{\hookrightarrow} \Lambda$. Assume that $\mathscr{F}$ and $\mathscr{F}|_{\Lambda'}$ are acyclic. If*

(i) *$i_*^0\colon H^0(\Lambda;\mathscr{A}^{\phi_*}) \to H^0(\Lambda';\mathscr{A}^{\phi_*})$ is injective, then im $i_*^0 \cong H^0(\Lambda;\mathscr{F})$ and*

$$H^1(\Lambda,\Lambda';\mathscr{F}) \cong \frac{H^0(\Lambda';\mathscr{F})}{im\ i_*^0}, \quad \textit{and} \quad H^k(\Lambda,\Lambda';\mathscr{F}) \cong 0, \quad \textit{for} \quad k \neq 1;$$

(ii) *$i_*^0\colon H^0(\Lambda;\mathscr{A}^{\phi_*}) \to H^0(\Lambda';\mathscr{A}^{\phi_*})$ is surjective, then*

$$H^0(\Lambda,\Lambda';\mathscr{F}) \cong \ker i_*^0, \quad \textit{and} \quad H^k(\Lambda,\Lambda';\mathscr{F}) \cong 0, \quad \textit{for} \quad k \neq 0.$$

Proof. As before for triple $(\Lambda', \emptyset) \stackrel{i}{\hookrightarrow} (\Lambda, \emptyset) \stackrel{j}{\hookrightarrow} (\Lambda, \Lambda')$ we have the long exact sequence,

$$0 \xrightarrow{\delta^0} H^0(\Lambda, \Lambda'; \mathscr{F}) \xrightarrow{j_*^0} H^0(\Lambda; \mathscr{F}) \xrightarrow{i_*^0} H^0(\Lambda'; \mathscr{F}) \xrightarrow{\delta^1} H^1(\Lambda, \Lambda'; \mathscr{F}) \xrightarrow{j_*^1} H^1(\Lambda; \mathscr{F}) \xrightarrow{i_*^1} H^1(\Lambda'; \mathscr{F}) \xrightarrow{\delta^2} \cdots.$$

Since, by Lemma 10.1, $\mathscr{F}$ is acyclic we obtain the truncated sequence

$$0 \xrightarrow{\delta^0} H^0(\Lambda, \Lambda'; \mathscr{F}) \xrightarrow{j_*^0} H^0(\Lambda; \mathscr{F}) \xrightarrow{i_*^0} H^0(\Lambda'; \mathscr{F}) \xrightarrow{\delta^1} H^1(\Lambda, \Lambda'; \mathscr{F}) \xrightarrow{j_*^1} 0. \tag{24}$$

Since i_*^0 is injective and thus $\ker i_*^0 = 0 = \operatorname{im} j_*^0$. Moreover, $\ker j_*^0 = \operatorname{im} \delta^0 = 0$, which implies that $H^0(\Lambda, \Lambda'; \mathscr{F}) \cong 0$. Consequently, we have the short exact sequence

$$0 \xrightarrow{j_*^0} H^0(\Lambda; \mathscr{F}) \xrightarrow{i_*^0} H^0(\Lambda'; \mathscr{F}) \xrightarrow{\delta^1} H^1(\Lambda, \Lambda'; \mathscr{F}) \xrightarrow{j_*^1} 0,$$

from which the result for $H^1(\Lambda, \Lambda'\mathscr{F})$ follows. The cohomology $H^k(\Lambda, \Lambda'\mathscr{F}) \cong 0$, for $k \geqslant 2$ follows from Lemma 9.12, which completes the proof of (i).

As for (ii) we have the truncated exact sequence in (24). Now i_*^0 is surjective which implies that $\ker \delta^1 = \operatorname{im} i_*^0 = H^0(\Lambda'; \mathscr{F})$. Therefore, $\ker j_*^1 = \operatorname{im} \delta^1 = 0$ and thus j_*^1 is injective. Consequently, $H^1(\Lambda, \Lambda'\mathscr{F}) \cong 0$. We now have the short exact sequence

$$0 \xrightarrow{\delta^0} H^0(\Lambda, \Lambda'; \mathscr{F}) \xrightarrow{j_*^0} H^0(\Lambda; \mathscr{F}) \xrightarrow{i_*^0} H^0(\Lambda'; \mathscr{F}) \xrightarrow{\delta^1} 0,$$

which implies that $H^0(\Lambda, \Lambda'; \mathscr{F}) \cong \ker i_*^0$. The relative homology for $k \geqslant 1$ follows from Lemma 9.12. □

Remark 10.3. The sheaf cohomology groups of the abelian attractor sheaf can be equipped with a cup product from the ring structure of the sheaf. We leave cup product computations and their interpretation for later work.

10.1.1. The pitchfork bifurcation

Consider a parametrized dynamical system on $X = \mathbb{R} \cup \{-\infty, \infty\}$, the 2-point compactification of $\mathbb{R}$, experiencing a pitchfork bifurcation, cf. Fig. 5. The parametrized flow is defined via the differential equation

$$\dot{x} = \lambda x - x^3, \quad x \in \mathbb{R}, \ \lambda \in \mathbb{R}.$$

For parameter values $\lambda \leqslant \lambda_0$, $\lambda_0 = 0$, there are two repelling fixed points at $+\infty$ and $-\infty$ and a single attracting fixed point at $x = 0$. However, if $\lambda > \lambda_0$ there are two attracting fixed points $x = \pm x_\lambda$, and $x = 0$ is instead a repelling fixed point. We fix a parametrization:

$$\boldsymbol{\psi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X),$$

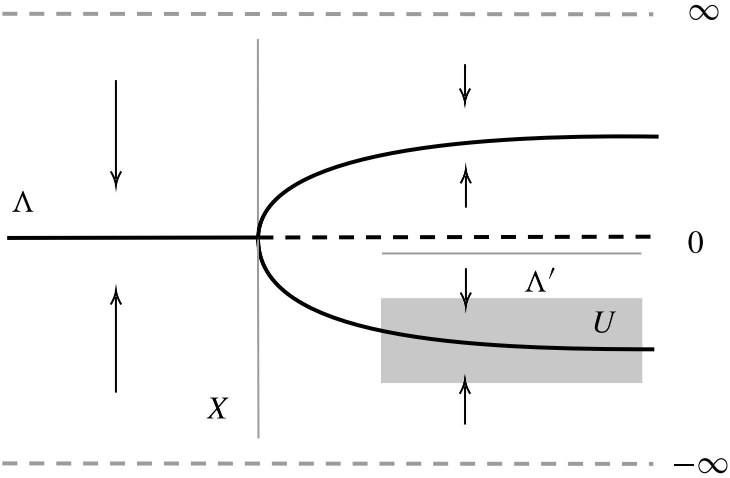


Fig. 5. In the pitchfork bifurcation, the section on $\Lambda' \subset \Lambda$ defined by $\sigma(\lambda) = \big(\lambda, \phi^\lambda, \omega_{\phi^\lambda}(U)\big)$ fails to extend globally.

where $\Lambda = \mathbb{R}$ is parameter space, $\mathbb{T} = \mathbb{R}^+$ is the time space and X is the 2-point compactification of $\mathbb{R}$.

Lemma 10.4. *The global sections of the abelian attractor sheaf for the normal form pitchfork bifurcation are* $H^0(\Lambda; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big) \cong \mathbb{Z}_2^3$.

Proof. Since $\boldsymbol{\psi}_*$ is stable everywhere except λ_0, Lemma 10.1 yields that $\Gamma(\mathscr{A}^{\boldsymbol{\psi}_*}) \cong \mathscr{A}^{\boldsymbol{\psi}_*}_{\lambda_0} \cong \mathsf{RAtt}(\boldsymbol{\psi}_*(\lambda_0))$. At λ_0, we have that $\mathsf{Att}(\boldsymbol{\psi}_*(\lambda_0))$ is a five element lattice: $\emptyset$, $\{0\}$, $[-\infty, 0]$, $[0, \infty]$, X. There are three join irreducible elements: $\{0\}$, $[-\infty, 0]$, and $[0, \infty]$. Applying $j\colon \mathsf{Att}(\boldsymbol{\psi}_*(\lambda_0)) \to \mathsf{RAtt}(\boldsymbol{\psi}_*(\lambda_0))$ to these yields the three generators of $\mathsf{RAtt}(\boldsymbol{\psi}_*(\lambda_0)) \cong \mathbb{Z}_2^3$. □

The above proof shows the global sections of $\mathscr{A}^{\boldsymbol{\psi}_*}$ are characterized by their value at the bifurcation point. The join-irreducible elements $\{0\}$, $[-\infty, 0]$, and $[0, \infty]$ for the stalk at λ_0 extend uniquely to the following global sections of $\boldsymbol{\psi}_*^{-1}\mathscr{S}^{\mathsf{Att}}$ (seen in purple in Fig. 6):

$$s_1 : \lambda \mapsto \begin{cases} (\lambda, \boldsymbol{\psi}_*(\lambda), \{0\}) & \lambda \leqslant 0 \\ (\lambda, \boldsymbol{\psi}_*(\lambda), [-x_\lambda, x_\lambda]) & \lambda > 0 \end{cases}, \qquad s_2 : \lambda \mapsto \begin{cases} (\lambda, \boldsymbol{\psi}_*(\lambda), [0, \infty]) & \lambda \leqslant 0 \\ (\lambda, \boldsymbol{\psi}_*(\lambda), [-\infty, x_\lambda]) & \lambda > 0 \end{cases},$$

$$s_3 : \lambda \mapsto \begin{cases} (\lambda, \boldsymbol{\psi}_*(\lambda), [-\infty, 0]) & \lambda \leqslant 0 \\ (\lambda, \boldsymbol{\psi}_*(\lambda), [-x_\lambda, \infty]) & \lambda > 0 \end{cases}.$$

Embedding these into $\Gamma(\mathscr{A}^{\boldsymbol{\psi}_*})$ via ℓ yields the canonical basis $\mathscr{B}_\Lambda$, as discussed at the end of Section 9.

Proposition 10.5. *Let* $\Lambda' := [a, \infty)$. *If* $a > 0$, *then* $H^k(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \mathbb{Z}_2^2$ *for* $k = 1$, *and* $H^k(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) = 0$ *otherwise. When* $a \leqslant 0$, *all relative cohomology groups vanish.*

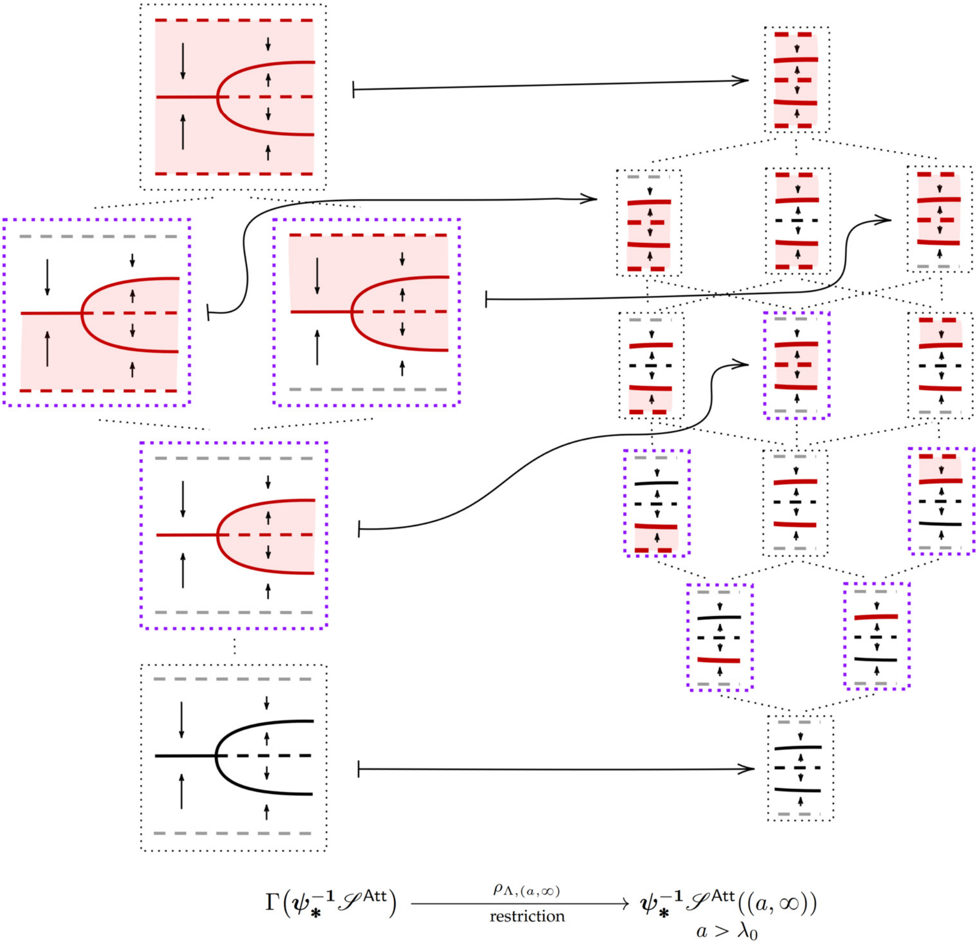


Fig. 6. Diagram of sections and restriction map for the pitchfork bifurcation's attractor lattice sheaf. Join-irreducible elements are highlighted in purple. Notice there are sections in $\psi_*^{-1}\mathscr{S}^{\mathsf{Att}}((a,b))$ which do not lie in the image of the restriction map.

Proof. Lemma 10.4 gives $\Gamma\left(\mathscr{A}^{\psi_*}\right) \cong \mathbb{Z}_2^3$. For $a > 0$, Λ' contains no bifurcation points, and so $\mathscr{A}^{\psi_*}\big|_{\Lambda'}$ is constant. By checking the attractor lattice at any stalk, a thirteen element lattice with five join-irreducible elements, we may obtain $H^0(\Lambda'; \mathscr{A}^{\psi_*}) \cong \Gamma\left(\mathscr{A}^{\psi_*}\big|_{\Lambda'}\right) \cong \mathscr{A}_\lambda^{\psi_*} \cong \mathbb{Z}_2^5$ for any $\lambda \in \Lambda'$. Now we consider the restriction map $i_*^0\colon H^0(\Lambda; \mathscr{A}^{\psi_*}) \to H^0(\Lambda'; \mathscr{A}^{\psi_*})$. We will write this map as a matrix in the canonical bases. Let $\{e_1, e_2, e_3\}$ denote the basis $\mathscr{B}_\Lambda$, where $e_i = \ell(s_i)$. $\{e_i'\}_{i=1}^5$, $e_i' = \ell(s_i')$ will denote the basis $\mathscr{B}_{\Lambda'}$, where

$$s_1' : \lambda \mapsto (\lambda, \psi_*(\lambda), \{-x_\lambda\}), \qquad s_2' : \lambda \mapsto (\lambda, \psi_*(\lambda), \{x_\lambda\}), \qquad s_3' : \lambda \mapsto (\lambda, \psi_*(\lambda), [-\infty, -x_\lambda]),$$

$$s_4' : \lambda \mapsto (\lambda, \psi_*(\lambda), [x_\lambda, \infty]), \qquad s_5' : \lambda \mapsto (\lambda, \psi_*(\lambda), [-x_\lambda, x_\lambda]).$$

Now the restriction map may be written as:

$$i_*^0 \colon H^0(\Lambda; \mathscr{A}^{\psi_*}) \to H^0(\Lambda'; \mathscr{A}^{\psi_*}), \qquad i_*^0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

i_*^0 is therefore injective, which by Lemma 10.2(i) yields $H^1(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) \cong \mathbb{Z}_2^2$, and that the other relative cohomology groups vanish. For $a \leqslant 0$, the argument of Lemma 10.1 shows that the restriction cohomomorphism induces an isomorphism on cohomology, so $H^0(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) = H^1(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) = 0$. Both $\mathscr{A}^{\psi_*}$ and $\mathscr{A}^{\psi_*}|_{\Lambda'}$ are acyclic, so the higher order relative cohomology groups vanish by Lemma 10.2(i). □

Proposition 10.6. *Let $\Lambda' := (-\infty, a]$. Then, $H^k(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) \cong 0$ for all k and for all $a \in \mathbb{R}$.*

Proof. Note that $\Gamma(\mathscr{A}^{\psi_*}) \cong \Gamma(\mathscr{A}^{\psi_*}|_{\Lambda'}) \cong \mathbb{Z}_2^3$ for all $a \in \mathbb{R}$ (the same computations as in Lemma 10.4 apply to Λ'). Therefore, $H^0(\Lambda; \mathscr{A}^{\psi_*}) \cong H^0(\Lambda'; \mathscr{A}^{\psi_*})$ for all $a \in \mathbb{R}$ and thus by Lemma 10.2(i) $H^k(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) \cong 0$ for all k. □

Theorem 10.7. *Let ϕ_* be a parametrized dynamical system over Λ conjugate to the above canonical parametrization ψ_* for the pitchfork bifurcation. Then,*

$$\mathscr{A}^{\phi_*} \quad \text{is acyclic and} \quad H^0(\Lambda; \mathscr{A}^{\phi_*}) \cong \mathbb{Z}_2^3.$$

Moreover, there exists a value $\lambda_0 \in \mathbb{R}$ such that

$$H^k(\Lambda, \Lambda'; \mathscr{A}^{\phi_*}) \cong \begin{cases} \mathbb{Z}_2^2 & \text{if } k = 1 \text{ and } a > \lambda_0; \\ 0 & \text{if } k \neq 1 \text{ or } a \leqslant \lambda_0, \end{cases}$$

where $\Lambda' = [a, \infty)$. Furthermore, for $\Lambda' := (-\infty, a]$, then $H^k(\Lambda, \Lambda'; \mathscr{A}^{\phi_*}) \cong 0$ for all k and for all $a \in \mathbb{R}$.

Proof. This follows immediately from Theorem 8.7, Lemma 10.1, and Propositions 10.5 and 10.6. □

This theorem can be applied locally in parameter space. If $\phi_* \colon \mathbb{R} \to \mathbf{DS}(\mathbb{R}; I)$ is some parametrized dynamical system such that ϕ_* experiences a pitchfork bifurcation on an open set Ω, then $\mathscr{A}^{\phi_*}|_\Omega$ has the above cohomology groups. Another important observation is that the relative cohomology in the example below is the same for a local pitchfork bifurcation.

Example 10.8. Let ϕ_* be a parametrized flow over $\Lambda = \mathbb{R}$ on the interval $X = [-1, 1]$ with a single attracting fixed point at $x = 0$ for $\lambda \leqslant 0$. This system is a semi-flow with $\mathbb{T} = \mathbb{R}^+$. For $\lambda \geqslant 0$ the system undergoes a pitchfork bifurcation with two branches $\pm x_\lambda$ of attracting fixed points converging to ± 1 respectively as $\lambda \to +\infty$, cf. Fig. 5. If we repeat the analysis in Propositions 10.5 and 10.6 the sheaf cohomology over Λ is different: $\mathscr{A}^{\phi_*}$ is acyclic and $H^0(\Lambda; \mathscr{A}^{\phi_*}) \cong \mathbb{Z}_2$. On the other hand the relative sheaf cohomologies $H^k(\mathbb{R}, [a, \infty); \mathscr{A}^{\phi_*})$ and $H^k(\mathbb{R}, (-\infty, a]; \mathscr{A}^{\phi_*})$ are isomorphic.

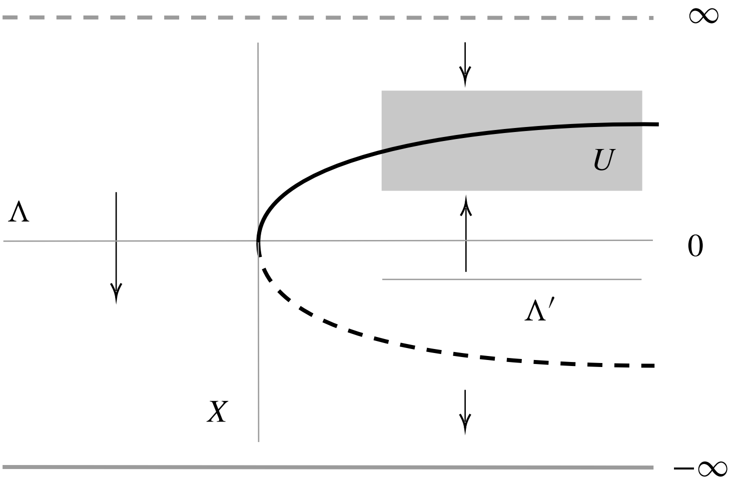


Fig. 7. A saddle-node bifurcation. The section on Λ' defined by $\sigma(\lambda) = \big(\lambda, \phi^\lambda, \omega_{\phi^\lambda}(U)\big)$ fails to extend globally.

10.1.2. The saddle-node bifurcation

Consider a parametrized dynamical system on $X = \mathbb{R} \cup \{-\infty, \infty\}$, the 2-point compactification of $\mathbb{R}$, experiencing a saddle-node bifurcation. The parametrized flow is defined via the differential equation

$$\dot{x} = \lambda - x^2, \quad x \in \mathbb{R},\ \lambda \in \mathbb{R},$$

and $+\infty$ and $-\infty$ are a repelling and attracting fixed point respectively. For parameter values less than $\lambda_0 := 0$, there is an attracting fixed point at $-\infty$, and a repelling fixed point at $+\infty$. For parameter values greater than λ_0, there is an additional attracting and repelling fixed point at x_λ and $-x_\lambda$ respectively (Figs. 7, 8). We fix a parametrization:

$$\boldsymbol{\psi}_* : \Lambda \to \mathbf{DS}(\mathbb{T}, X),$$

where $\Lambda = \mathbb{R}$ is parameter space, $\mathbb{T} = \mathbb{R}$ is the time space and X is the 2-point compactification of $\mathbb{R}$. Lemma 10.1 again shows that the attractor sheaf $\mathscr{A}^{\boldsymbol{\psi}_*}$ has vanishing higher order cohomology, but relative cohomology recognizes the bifurcations.

Lemma 10.9. *The global sections of the abelian attractor sheaf for the normal form saddle-node bifurcation are* $H^0(\Lambda; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big) \cong \mathbb{Z}_2^3$.

Proof. We follow a similar proof to 10.4. $\boldsymbol{\psi}_*$ is stable everywhere except λ_0, so $\Gamma(\mathscr{A}^{\boldsymbol{\psi}_*}) \cong \mathsf{RAtt}(\boldsymbol{\psi}_*(\lambda_0))$. At λ_0, we have that $\mathsf{Att}(\boldsymbol{\psi}_*(\lambda_0))$ is a four element lattice: $\emptyset$, $\{-\infty\}$, $[-\infty, 0]$, X. There are three join irreducible elements: $\{-\infty\}$, $[-\infty, 0]$, and X. Applying $j\colon \mathsf{Att}(\boldsymbol{\psi}_*(\lambda_0)) \to \mathsf{RAtt}(\boldsymbol{\psi}_*(\lambda_0))$ to these yields the three generators of $\mathsf{RAtt}(\boldsymbol{\psi}_*(\lambda_0)) \cong \mathbb{Z}_2^3$. $\square$

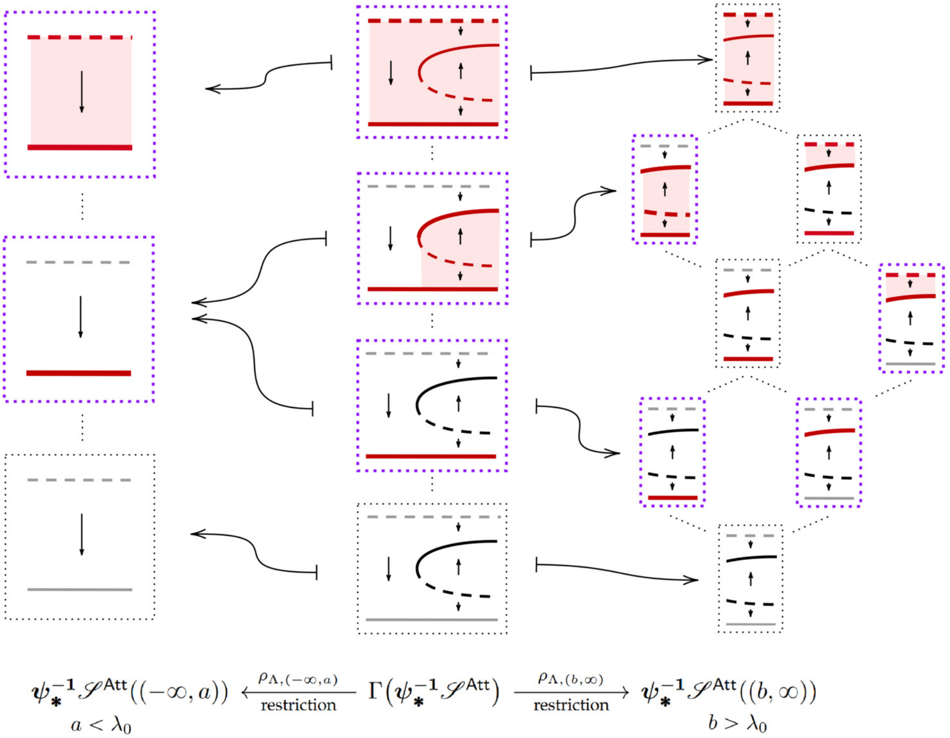


Fig. 8. Diagram of sections and restriction maps for the saddle-node bifurcation's attractor lattice sheaf. Join-irreducible elements are highlighted in purple. This example shows restriction maps can fail to be injective or surjective.

One can explicitly calculate the join-irreducible elements of $\Gamma(\psi_*^{-1}\mathscr{S}^{\mathsf{Att}})$:

$$s_1 : \lambda \mapsto (\lambda, \psi_*(\lambda), \{-\infty\}), \quad s_2 : \lambda \mapsto \begin{cases} (\lambda, \psi_*(\lambda), \{-\infty\}) & \lambda < \lambda_0 \\ (\lambda, \psi_*(\lambda), [-\infty, x_\lambda]) & \lambda \geqslant \lambda_0 \end{cases},$$
$$s_3 : \lambda \mapsto (\lambda, \psi_*(\lambda), X).$$

As with the pitchfork bifurcation, embedding these into $\Gamma(\mathscr{A}^{\psi_*})$ via ℓ yields the canonical basis $\mathscr{B}_\Lambda$, cf. Section 9.

Proposition 10.10. *Let $\Lambda' = [a, \infty)$. If $a > 0$, then $H^k(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) \cong \mathbb{Z}_2$ for $k = 1$, and vanishes otherwise. When $a \leqslant 0$, then $H^k(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) = 0$ for all k.*

Proof. Lemma 10.9 demonstratesthe global sections are $H^0(\Lambda; \mathscr{A}^{\psi_*}) \cong \Gamma\big(\mathscr{A}^{\psi_*}\big) \cong \mathbb{Z}_2^3$. For $a > 0$, Λ' contains no bifurcation points. So we may check at any stalk $\lambda \in \Lambda'$ that

$$\mathsf{RAtt}(\psi_*(\lambda_0)) \cong H^0(\Lambda'; \mathscr{A}^{\psi_*}) \cong \Gamma\big(\mathscr{A}^{\psi_*}\big|_{\Lambda'}\big) \cong \mathbb{Z}_2^4.$$

Now we must consider the restriction map i_*^0. Let $\{e_1, e_2, e_3\}$ the basis $\mathscr{B}_\Lambda$, where $e_i = \ell(s_i)$. Define

$$s_1'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), \{-\infty\}), \quad s_2'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), \{x_\lambda\}),$$
$$s_3'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [x_\lambda, \infty]), \quad s_4'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [-\infty, x_\lambda]).$$

The canonical basis $\mathscr{B}_{\Lambda'}$ is given by $\{e_i'\}_{i=1}^4$, where $e_i' = \ell(s_i')$. i_*^0 may be written as a matrix in these bases:

$$i_*^0\colon H^0(\Lambda; \mathscr{A}^{\boldsymbol{\psi}_*}) \to H^0(\Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}), \quad i_*^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The injectivity of i_*^0 and Lemma 10.2(i) yields $H^1(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \mathbb{Z}_2$, and the other relative cohomology groups vanish. For $a \leqslant 0$, Lemma 10.1 proves $H^0(\Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda'}\big) \cong \mathbb{Z}_2^3$, so $H^0(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) = H^1(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) = 0$ since the restriction is an isomorphism. As before the higher order relative cohomology groups vanish by Lemma 10.2(i). □

Proposition 10.11. *Let $\Lambda' = (-\infty, a]$. If $a \geqslant 0$, then $H^k(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong 0$ for all k. When $a < 0$, then $H^0(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \mathbb{Z}_2$ and vanishes otherwise.*

Proof. As before the global sections are $H^0(\Lambda; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big) \cong \mathbb{Z}_2^3$. For $a \geqslant 0$, we have $H^0(\Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda'}\big) \cong \mathbb{Z}_2^3$ by applying 10.1. The restriction map i_*^0 is an isomorphism, so all relative cohomology groups vanish. For $a < 0$, there are no bifurcation points in Λ', and so we have $H^0(\Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda'}\big) \cong \mathbb{Z}_2^2$ by checking $\mathsf{RAtt}(\boldsymbol{\psi}_*(\lambda))$ at any stalk in Λ'. Define

$$s_1'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), \{-\infty\}), \quad s_2'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), X).$$

The canonical basis $\mathscr{B}_{\Lambda'}$ is then given by $e_i' := \ell(s_i')$. Thus, in the canonical bases,

$$i_*^0\colon H^0(\Lambda; \mathscr{A}^{\boldsymbol{\psi}_*}) \to H^0(\Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}), \quad i_*^0 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Surjectivity of i_*^0 and Lemma 10.2(ii) then implythat $H^0(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \mathbb{Z}_2$. The higher order relative cohomology groups vanish by Lemma 10.2(i) and (ii). □

Theorem 10.12. *Let $\boldsymbol{\phi}_*$ be a parametrized dynamical system over Λ conjugate to the above canonical parametrization $\boldsymbol{\psi}_*$ for the saddle-node bifurcation. Then,*

$$\mathscr{A}^{\boldsymbol{\phi}_*} \quad \textit{is acyclic and} \quad H^0(\Lambda; \mathscr{A}^{\boldsymbol{\phi}_*}) \cong \mathbb{Z}_2^3.$$

Moreover, there exists a value $\lambda_0 \in \mathbb{R}$ such that

$$H^k(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\phi}_*}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 1 \text{ and } a > \lambda_0 \\ 0 & k \neq 1, \text{ or } a \leqslant \lambda_0, \end{cases} \quad \text{with} \quad \Lambda' = [a, \infty),$$

$$H^k(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\phi}_*}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 0 \text{ and } a < \lambda_0 \\ 0 & k \neq 0, \text{ or } a \geqslant \lambda_0, \end{cases} \quad \text{with} \quad \Lambda' = (-\infty, a].$$

Proof. Apply Theorem 8.7, Lemma 10.1, and Propositions 10.10 and 10.11. □

Remark 10.13. Fix $a < 0$. Let us more closely examine the restriction map $i^0_*\colon H^0(\Lambda; \mathscr{A}^{\boldsymbol{\psi}_*}) \to H^0((-\infty, a]; \mathscr{A}^{\boldsymbol{\psi}_*})$. $\ker i^0_*$, which we showed is isomorphic to $H^0\big(\mathbb{R}, (-\infty, a]; \mathscr{A}^{\boldsymbol{\psi}_*}\big)$, is generated by $e_2 + e_3 \in H^0(\Lambda; \mathscr{A}^{\boldsymbol{\psi}_*})$. $e_2 + e_3 \neq 0$, but $i^0_*(e_2) = i^0_*(e_3) = e'_1$. We can compute the symmetric Conley form on the sections s_2, s_3:

$$\boldsymbol{\psi}_*^{-1}\mathscr{S}^{C_{\mathsf{Att}}}(s_3 \vee s_2, s_3 \wedge s_2) = \boldsymbol{\psi}_*^{-1}\mathscr{S}^{C_{\mathsf{Att}}}(s_3, s_2) = g \in \Gamma(\boldsymbol{\psi}_*^{-1}\mathscr{S}^{\mathsf{Morse}}),$$

$$g : \lambda \mapsto \begin{cases} (\lambda, \boldsymbol{\psi}_*(\lambda), \emptyset) & \lambda < \lambda_0 \\ (\lambda, \boldsymbol{\psi}_*(\lambda), [-x_\lambda, x_\lambda]) & \lambda \geqslant \lambda_0 \end{cases}.$$

Just as $e_2 + e_3$ is nonzero only on $[0, \infty)$, g is nonempty only on $[0, \infty)$. Continuation of a Morse set to the empty set via the global section g yields nontrivial relative cohomology. We leave the rigorous formulation of this (utilizing a map from $\Gamma(\boldsymbol{\psi}_*^{-1}\mathscr{S}^{\mathsf{Morse}})$ to $\Gamma(\mathscr{A}^{\boldsymbol{\psi}_*})$ similar in spirit to ℓ) for later work.

Example 10.14. Consider a saddle-node bifurcation in the system described in Fig. 7. We impose an attracting fixed point at the bottom of Fig. 7, such that we may restrict phase space to a forward-invariant compact interval $X = [1, 0]$. Call this parametrized dynamical system $\boldsymbol{\phi}_*\colon \mathbb{R} \to \mathbf{DS}(\mathbb{R}^+, X)$. Lemma 10.1 again shows that $\mathscr{A}^{\boldsymbol{\phi}_*}$ has vanishing higher cohomology. However, $H^0(\Lambda; \mathscr{A}^{\boldsymbol{\phi}_*}) \cong \mathbb{Z}_2^2$ which differs from the above example. The relative cohomology groups are the same as in the above example as is the case for the pitchfork bifurcation.

10.1.3. The transcritical bifurcation

Consider a parametrized dynamical system on $X = \mathbb{R} \cup \{-\infty, \infty\}$, the 2-point compactification of $\mathbb{R}$, experiencing a transcritical bifurcation. The parametrized flow is defined via the differential equation

$$\dot{x} = \lambda x - x^2, \quad x \in \mathbb{R},\ \lambda \in \mathbb{R},$$

and $+\infty$ and $-\infty$ are a repelling and attracting fixed points respectively. As before we fix a parametrization:

$$\boldsymbol{\psi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X),$$

where $\Lambda = \mathbb{R}$ is parameter space, $\mathbb{T} = \mathbb{R}$ is the time space and X is the 2-point compactification of $\mathbb{R}$. Lemma 10.1 again shows that the attractor sheaf $\mathscr{A}^{\boldsymbol{\psi}_*}$ has vanishing higher order cohomology. We denote $\lambda_0 := 0$ the bifurcation point, and $y_\lambda < x_\lambda$ denote the two roots of $\lambda x - x^2$. (Figs. 9, 10.)

Lemma 10.15. *The global sections of the abelian attractor sheaf for the normal form transcritical bifurcation are* $H^0(\Lambda; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big) \cong \mathbb{Z}_2^3$.

Proof. Similar proof to 10.4 and 10.9. $\boldsymbol{\psi}_*$ is stable everywhere except λ_0, so $\Gamma(\mathscr{A}^{\boldsymbol{\psi}_*}) \cong \mathsf{RAtt}(\boldsymbol{\psi}_*(\lambda_0))$. At λ_0, we have that $\mathsf{Att}(\boldsymbol{\psi}_*(\lambda_0))$ is a four element lattice: $\emptyset$, $\{-\infty\}$, $[-\infty, 0]$, X.

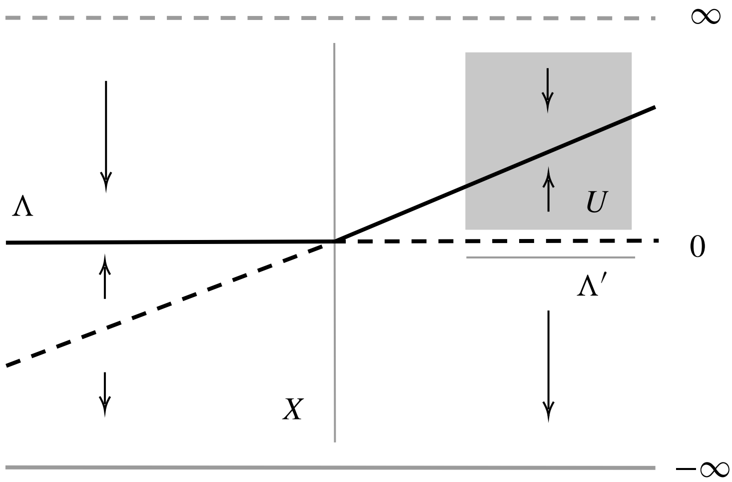


Fig. 9. A trans-critical bifurcation. The section on Λ' defined by $\sigma(\lambda) = \big(\lambda, \phi^\lambda, \omega_{\phi^\lambda}(U)\big)$ fails to extend globally.

There are three join irreducible elements: $\{-\infty\}$, $[-\infty, 0]$, and X. Applying $j\colon \mathsf{Att}(\boldsymbol{\psi}_*(\lambda_0)) \to \mathsf{RAtt}(\boldsymbol{\psi}_*(\lambda_0))$ to these yields the three generators of $\mathsf{RAtt}(\boldsymbol{\psi}_*(\lambda_0)) \cong \mathbb{Z}_2^3$. $\square$

Computing the join-irreducible elements of $\Gamma(\boldsymbol{\psi}_*^{-1}\mathscr{S}^{\mathsf{Att}})$:

$$s_1\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), \{-\infty\}), \quad s_2\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [-\infty, x_\lambda]),$$

$$s_3\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), X).$$

Embedding these into $\Gamma(\mathscr{A}^{\boldsymbol{\psi}_*})$ via ℓ yields the canonical basis $\mathscr{B}_\Lambda$, cf. Section 9.

Proposition 10.16. *Let $\Lambda' = [a, \infty)$. If $a > 0$, then $H^k(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \mathbb{Z}_2$ for $k = 1$, and vanishes otherwise. When $a \leqslant 0$, then $H^k(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) = 0$ for all k.*

Proof. By 10.15, the global sections are $H^0(\Lambda; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big) \cong \mathbb{Z}_2^3$. For $a > 0$, Λ' contains no bifurcation points. Thus, $\mathscr{A}\big|_{\Lambda'}$ is constant on Λ'. We have

$$H^0(\Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \Gamma\Big(\mathscr{A}^{\boldsymbol{\psi}_*}\Big|_{\Lambda'}\Big) \cong \mathsf{RAtt}(\boldsymbol{\psi}_*(\lambda)) \cong \mathbb{Z}_2^4,$$

for any $\lambda \in \Lambda'$. Now for i_*^0. Let $\{e_1, e_2, e_3\}$ be the basis B_Λ, where $e_i = \ell(s_i)$. We define

$$s_1'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), \{-\infty\}), \quad s_2'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), \{x_\lambda\}),$$

$$s_3'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [x_\lambda, \infty]), \quad s_4'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [-\infty, x_\lambda]).$$

$\mathscr{B}_{\Lambda'}$ is given by $\{e_i'\}_{i=1}^4$, where $e_i' = \ell(s_i')$. i_*^0 may be written as a matrix in these bases:

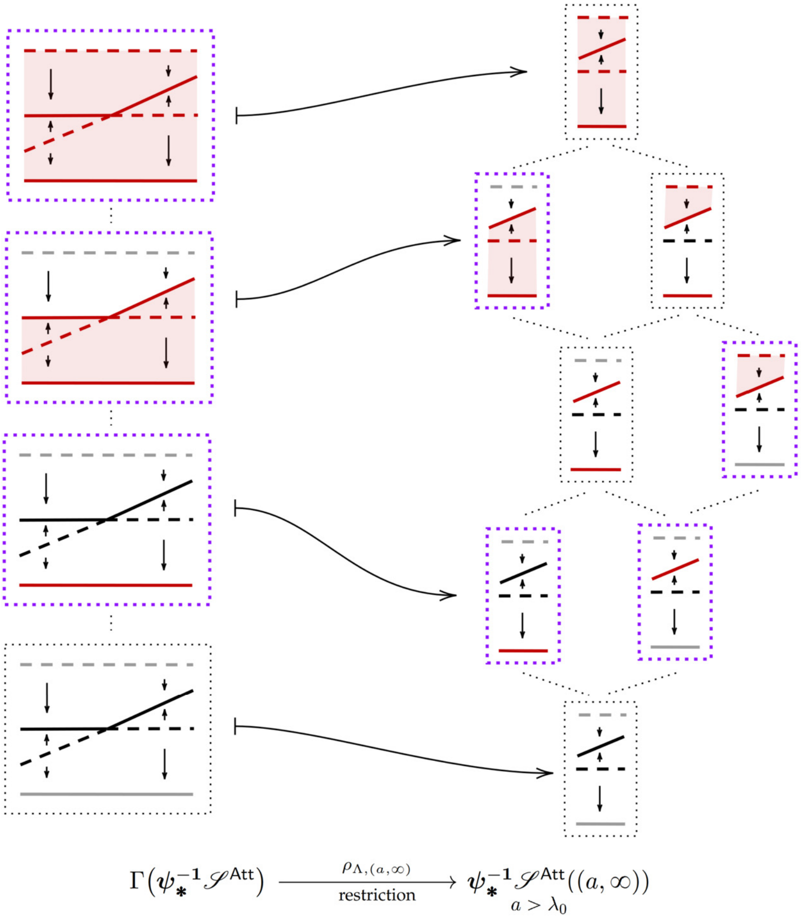


Fig. 10. Diagram of sections and restriction map for the transcritical bifurcation's attractor lattice sheaf. Join-irreducible elements are highlighted in purple.

$$i_*^0 \colon H^0(\Lambda; \mathscr{A}^{\psi_*}) \to H^0(\Lambda'; \mathscr{A}^{\psi_*}), \quad i_*^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The injectivity of i_*^0 and Lemma 10.2(i) yields $H^1(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) \cong \mathbb{Z}_2$. For $a \leqslant 0$, Lemma 10.1 proves we have $H^0(\Lambda'; \mathscr{A}^{\psi_*}) \cong \Gamma\big(\mathscr{A}^{\psi_*}\big|_{\Lambda'}\big) \cong \mathbb{Z}_2^3$. Since the restriction is an isomorphism, all relative cohomology groups vanish. □

Proposition 10.17. *Let $\Lambda' = (-\infty, a]$. If $a \geqslant 0$, then $H^k(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) \cong 0$ for all k. When $a < 0$, then $H^1(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) \cong \mathbb{Z}_2$ and vanishes otherwise.*

Proof. The proof proceeds mirror to Proposition 10.16. For $a \geqslant 0$, we have $H^0(\Lambda'; \mathscr{A}^{\psi_*}) \cong \Gamma\big(\mathscr{A}^{\psi_*}\big|_{\Lambda'}\big) \cong \mathbb{Z}_2^3$. The restriction is an isomorphism, yielding that all relative cohomology groups vanish. For $a < 0$, we have $H^0(\Lambda'; \mathscr{A}^{\psi_*}) \cong \Gamma\big(\mathscr{A}^{\psi_*}|_{\Lambda'}\big) \cong \mathbb{Z}_2^4$. Let $\{e_1, e_2, e_3\}$ be the basis B_Λ, where $e_i = \ell(s_i)$. We define

$$s'_1 \colon \lambda \mapsto (\lambda, \psi_*(\lambda), \{-\infty\}), \quad s'_2 \colon \lambda \mapsto (\lambda, \psi_*(\lambda), \{x_\lambda\}),$$
$$s'_3 \colon \lambda \mapsto (\lambda, \psi_*(\lambda), [x_\lambda, \infty]), \quad s'_4 \colon \lambda \mapsto (\lambda, \psi_*(\lambda), [-\infty, x_\lambda]).$$

$B_{\Lambda'}$ is given by $\{e'_i\}_{i=1}^4$, where $e'_i = \ell(s'_i)$. In these bases,

$$i_*^0 \colon H^0(\Lambda; \mathscr{A}^{\psi_*}) \to H^0(\Lambda'; \mathscr{A}^{\psi_*}), \quad i_*^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The injectivity of i_*^0 and Lemma 10.2(i) then implies that $H^1(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) \cong \mathbb{Z}_2$. The other relative cohomology groups vanish by Lemma 10.2(i) also. □

Theorem 10.18. *Let ϕ_* be a parametrized dynamical system over Λ conjugate to the above canonical parametrization for the transcritical bifurcation. Then,*

$$\mathscr{A}^{\phi_*} \quad \textit{is acyclic and} \quad H^0(\Lambda; \mathscr{A}^{\phi_*}) \cong \mathbb{Z}_2^3.$$

Moreover, there exists a value $\lambda_0 \in \mathbb{R}$ such that

$$H^k(\Lambda, \Lambda'; \mathscr{A}^{\phi_*}) \cong \begin{cases} \mathbb{Z}_2 & \textit{if } k = 1 \textit{ and } a > \lambda_0 \\ 0 & k \neq 1, \textit{or } a \leqslant \lambda_0, \end{cases} \quad \textit{with} \quad \Lambda' = [a, \infty),$$

$$H^k(\Lambda, \Lambda'; \mathscr{A}^{\phi_*}) \cong \begin{cases} \mathbb{Z}_2 & \textit{if } k = 1 \textit{ and } a < \lambda_0 \\ 0 & k \neq 1, \textit{or } a \geqslant \lambda_0, \end{cases} \quad \textit{with} \quad \Lambda' = (-\infty, a].$$

Proof. Apply Theorem 8.7, Lemma 10.1 and Propositions 10.10 and 10.11. □

Remark 10.19. Note the subtle difference in the relative sheaf cohomology for the saddle-node and transcritical bifurcations. For the latter we only find relative cohomology at $k = 1$ for different choices of Λ', as for the saddle-node we have cohomology at $k = 0$ and $k = 1$ for various choices of Λ'.

10.2. The S-shaped bifurcation

Now we study the S-shaped bifurcation, as in Fig. 11. Note this bifurcation has two critical parameter values. Consider a parametrized dynamical system on $X = \mathbb{R} \cup \{-\infty, \infty\}$, the 2-point compactification of $\mathbb{R}$, experiencing an S-shaped bifurcation. The parametrized flow is defined via the differential equation

$$\dot{x} = \lambda + x - x^3, \quad x \in \mathbb{R}, \ \lambda \in \mathbb{R},$$

and $+\infty$ and $-\infty$ are a repelling fixed points. As before we fix a parametrization:

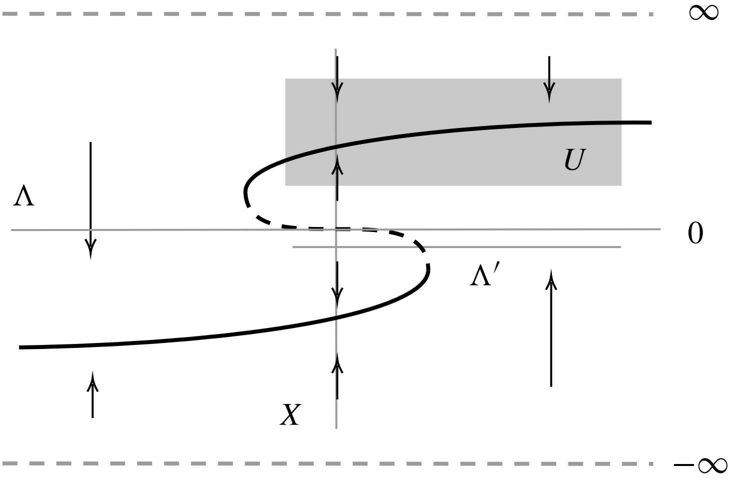


Fig. 11. An S-shaped bifurcation. The section on Λ' defined by $\sigma(\lambda) = \big(\lambda, \phi^\lambda, \omega_{\phi^\lambda}(U)\big)$ fails to extend globally.

$$\boldsymbol{\psi}_*\colon \Lambda \to \mathbf{DS}(\mathbb{T}, X),$$

where $\Lambda = \mathbb{R}$ is parameter space, $\mathbb{T} = \mathbb{R}$ is the time space and X is the 2-point compactification of $\mathbb{R}$. Here, there are two bifurcation points at $\lambda_1 := -1$ and $\lambda_2 := +1$. We let $y_\lambda \leqslant x_\lambda$ denote the least and greatest root of $\lambda + x - x^3$, respectively (potentially coinciding).

Proposition 10.20. *$\mathscr{A}^{\boldsymbol{\psi}_*}$ is acyclic, and $\Gamma(\mathscr{A}^{\boldsymbol{\psi}_*}) \cong \mathbb{Z}_2^3$.*

Proof. Pick $\lambda_1 < a < b < \lambda_2$, such that $\Lambda_1 := (-\infty, b]$ and $\Lambda_2 := [a, \infty)$ cover $\mathbb{R}$. Consider the Mayer-Vietoris exact sequence:

$$0 \xrightarrow{\delta^0} \Gamma(\mathscr{A}^{\boldsymbol{\psi}_*}) \xrightarrow{\alpha^0_*} \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_1}\big) \oplus \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_2}\big) \xrightarrow{\beta^0_*} \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{[a,b]}\big)$$

$$\xrightarrow{\delta^1} H^1(\Lambda; \mathscr{A}^{\boldsymbol{\psi}_*}) \xrightarrow{\alpha^1_*} H^1\big(\Lambda_1; \mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_1}\big) \oplus H^1\big(\Lambda_2; \mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_2}\big) \xrightarrow{\beta^1_*} H^1\big([a,b]; \mathscr{A}^{\boldsymbol{\psi}_*}\big|_{[a,b]}\big) \to 0,$$

since $H^2(\mathbb{R}; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong 0$, which uses the fact that intervals have covering dimension 1, cf. [48, Lemma 2.7.3 and Proposition 3.2.2]. We can compute the global sections:

$$\Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_1}\big) \cong \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_2}\big) \cong \mathbb{Z}_2^4, \qquad \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{[a,b]}\big) \cong \mathbb{Z}_2^5.$$

The computations for $\Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}|_{\Lambda_1}\big)$ and $\Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}|_{\Lambda_2}\big)$ are identical, both employing 10.1 to achieve an isomorphism between the global sections and the stalk at the bifurcation point. As for $\Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}|_{[a,b]}\big)$, there are no bifurcation points in $[a, b]$, so we may check this at any stalk in $[a, b]$. Now we will write some bases. For $\Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}|_{\Lambda_1}\big)$, the join irreducible elements are

$$s_1'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), \{y_\lambda\}), \quad s_2'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [y_\lambda, x_\lambda]),$$

$$s_3'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [-\infty, y_\lambda]), \quad s_4'\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [y_\lambda, \infty]).$$

The canonical basis $\mathscr{B}_{\Lambda_1}$ is then $\{e_i'\}_{i=1}^4$ where $e_i' = \ell(s_i')$. As for $\Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_2}\big)$,

$$s_1''\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), \{x_\lambda\}), \quad s_2''\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [x_\lambda, \infty]),$$
$$s_3''\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [y_\lambda, x_\lambda]), \quad s_4''\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [-\infty, x_\lambda]).$$

So $\mathscr{B}_{\Lambda_2}$ is then $\{e_i''\}_{i=1}^4$ where $e_i'' = \ell(s_i'')$. Finally, on $\Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{[a,b]}\big)$:

$$\widehat{s}_1 : \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), \{y_\lambda\}), \quad \widehat{s}_2 : \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), \{x_\lambda\}), \quad \widehat{s}_3 : \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [-\infty, y_\lambda]),$$
$$\widehat{s}_4 : \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [x_\lambda, \infty]), \quad \widehat{s}_5 : \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), [y_\lambda, x_\lambda]).$$

Hence $\mathscr{B}_{\Lambda_1} = \{\widehat{e}_i\}_{i=1}^5$ where $\widehat{e}_i = \ell(\widehat{s}_i)$. The restriction maps $\rho_i\colon \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_i}\big) \to \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{[a,b]}\big)$ can now be represented:

$$\rho_1\colon \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_i}\big) \to \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{[a,b]}\big), \quad \rho_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\rho_2\colon \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_i}\big) \to \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{[a,b]}\big), \quad \rho_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

So we may write $\beta_*^0 = \rho_1 - \rho_2 = \rho_1 + \rho_2$:

$$\beta_*^0\colon \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_1}\big) \oplus \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_2}\big) \to \Gamma\big(\mathscr{A}^{\boldsymbol{\psi}_*}\big|_{[a,b]}\big), \quad \beta_*^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

So $\Gamma(\mathscr{A}^{\boldsymbol{\psi}_*}) \cong \operatorname{im} \alpha_*^0 \cong \ker \beta_*^0 \cong \mathbb{Z}_2^3$. Since Λ_1, Λ_2 both contain only one bifurcation point, we can apply Lemma 10.1 to conclude that $H^1\big(\Lambda_1; \mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_1}\big)$ and $H^1\big(\Lambda_2; \mathscr{A}^{\boldsymbol{\psi}_*}\big|_{\Lambda_2}\big)$ vanish for all $k \geqslant 1$. As β_*^0 is surjective, this implies $H^1(\Lambda; \mathscr{A}^{\boldsymbol{\psi}_*}) = 0$. The remaining sheaf cohomology vanishes due to the dimension restriction on Λ. □

The S-shaped bifurcation is an example where $\mathscr{A}^{\boldsymbol{\psi}_*}$ and $\mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\psi}_*}$, the attractor sheaf and free attractor sheaf respectively, have differing cohomologies.

Proposition 10.21. *$\mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\psi}_*}((-\infty, \lambda_2)) \cong \mathbb{Z}_2^7$, $\mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\psi}_*}((\lambda_1, \infty)) \cong \mathbb{Z}_2^7$, and $\mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\psi}_*}((\lambda_1, \lambda_2)) \cong \mathbb{Z}_2^{13}$.*

Proof. Both $(-\infty, \lambda_2)$ and (λ_1, ∞) contain precisely one bifurcation point, so we may apply Lemma 10.1 to get isomorphisms $\mathscr{A}tt^{\psi_*}((-\infty, \lambda_2)) \cong \mathscr{A}tt^{\psi_*}_{\lambda_1}$ and $\mathscr{A}tt^{\psi_*}((\lambda_1, \infty)) \cong \mathscr{A}tt^{\psi_*}_{\lambda_2}$. Note that $\mathsf{Att}(\boldsymbol{\psi}_*(\lambda_1))$ and $\mathsf{Att}(\boldsymbol{\psi}_*(\lambda_2))$ are both seven element lattices. Thus, $\mathscr{A}tt^{\psi_*}_{\lambda_1} \cong \mathbb{Z}_2^7$ and $\mathscr{A}tt^{\psi_*}_{\lambda_2} \cong \mathbb{Z}_2^7$, the free monoid rings generated by these lattices. As for $\mathscr{A}tt^{\psi_*}((\lambda_1, \lambda_2))$, (λ_1, λ_2) contains no bifurcation points, so $\mathscr{A}tt^{\psi_*}((\lambda_1, \lambda_2)) \cong \mathscr{A}tt^{\psi_*}_{\lambda}$ for any $\lambda \in (\lambda_1, \lambda_2)$. At any such λ, $\mathsf{Att}(\boldsymbol{\psi}_*(\lambda))$ is a thirteen element lattice. So $\mathscr{A}tt^{\psi_*}((\lambda_1, \lambda_2)) \cong \mathbb{Z}_2^{13}$. $\square$

Proposition 10.22. *Let $\mathscr{A}tt^{\psi_*}$ be the free attractor sheaf associated to $\boldsymbol{\psi}_*$.*

$$H^k(\Lambda; \mathscr{A}tt^{\psi_*}) \cong \begin{cases} \mathbb{Z}_2^5 & \text{if } k = 0 \\ \mathbb{Z}_2^4 & \text{if } k = 1 \\ 0 & \text{if } k \geqslant 2 \end{cases}$$

Proof. Let $\Lambda_1 = (-\infty, \lambda_2)$ and $\Lambda_2 = (\lambda_1, \infty)$ be an open covering for $\Lambda = \mathbb{R}$. We build the *ordered Čech complex* from this cover:

$$0 \longrightarrow \overline{C}^0(\{\Lambda_1, \Lambda_2\}; \mathscr{A}tt^{\psi_*}) \xrightarrow{\delta^0} \overline{C}^1(\{\Lambda_1, \Lambda_2\}; \mathscr{A}tt^{\psi_*}) \xrightarrow{\delta^1} \overline{C}^2(\{\Lambda_1, \Lambda_2\}; \mathscr{A}tt^{\psi_*}) \xrightarrow{\delta^2} \dots,$$

which in our case is:

$$0 \longrightarrow \mathscr{A}tt^{\psi_*}(\Lambda_1) \oplus \mathscr{A}tt^{\psi_*}(\Lambda_2) \xrightarrow{\rho^2_{1,2} - \rho^1_{1,2}} \mathscr{A}tt^{\psi_*}(\Lambda_1 \cap \Lambda_2) \longrightarrow 0,$$

where $\rho^2_{1,2}$ denotes the restriction map from $\mathscr{A}tt^{\psi_*}(\Lambda_1)$ to $\mathscr{A}tt^{\psi_*}(\Lambda_1 \cap \Lambda_2)$, and $\rho^1_{1,2}$ from $\mathscr{A}tt^{\psi_*}(\Lambda_2)$. We can write these as matrices:

$$\rho^2_{1,2} = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}, \quad \rho^1_{1,2} = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}$$

where the bases are read from the corresponding generators in Fig. 12, bottom up, left to right. There are four generators for $\mathscr{A}tt^{\psi_*}(\Lambda_1 \cap \Lambda_2)$ that do not lie in the image of either restriction map. We get cohomology groups from the above chain complex:

$$\check{H}^0(\{\Lambda_1, \Lambda_2\}; \mathscr{A}tt^{\psi_*}) = \ker \delta_1 \cong \mathbb{Z}_2^5, \qquad \check{H}^1(\{\Lambda_1, \Lambda_2\}; \mathscr{A}tt^{\psi_*}) = \ker \delta_2 / \mathrm{Im} \delta_1 \cong \mathbb{Z}_2^4,$$
$$\check{H}^k(\{\Lambda_1, \Lambda_2\}; \mathscr{A}tt^{\psi_*}) = 0 \text{ for } k > 1.$$

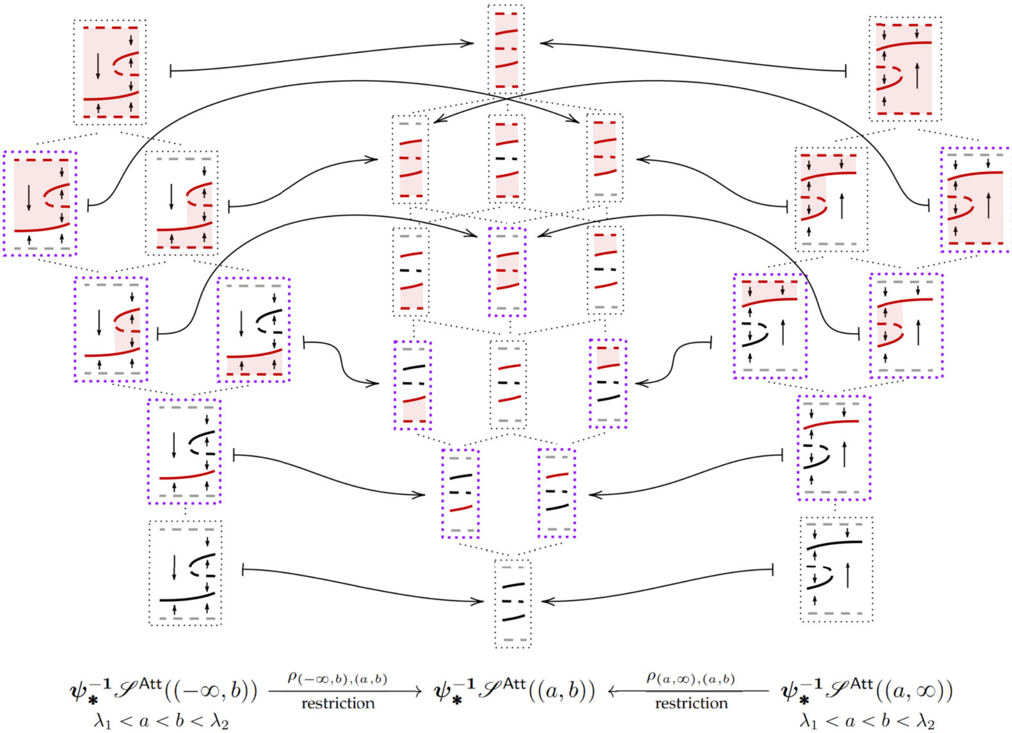


Fig. 12. Diagram of sections and restriction maps for the S-shaped bifurcation's attractor lattice sheaf. Join-irreducible elements are highlighted in purple. Notice that while there are sections in $\boldsymbol{\psi}_*^{-1}\mathscr{S}^{\mathsf{Att}}((a,b))$ which do not lie in the image of either restriction map, the join-irreducible elements of $\boldsymbol{\psi}_*^{-1}\mathscr{S}^{\mathsf{Att}}((a,b))$ do. We observe a consequence of this in Section 10.3: $\mathscr{A}^{\boldsymbol{\psi}_*}$ is acyclic while $\mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\psi}_*}$ is not.

Since $\Lambda_1 \cap \Lambda_2$ contains no bifurcation points, $\mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\psi}_*}$ is locally constant on $\Lambda_1 \cap \Lambda_2$ and therefore acyclic on $\Lambda_1 \cap \Lambda_2$. We now use Leray's Theorem to determine the sheaf cohomology of $\mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\psi}_*}$ from the above Čech cohomology groups, which yields the desired result. □

The result of Proposition 10.22 is an example where Theorem 9.11 applies. The sheaf cohomology of $\mathscr{A}\!\mathscr{t}\!\mathscr{t}^{\boldsymbol{\psi}_*}$ picks up bifurcations. For the sheaf cohomology of $\mathscr{A}^{\boldsymbol{\psi}_*}$ Theorem 9.11 does not apply.

Proposition 10.23. *Let $\Lambda' = [a, \infty)$. If $a \in (\lambda_1, \lambda_2]$, then $H^k(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong \mathbb{Z}_2$ for $k = 1$, and vanishes otherwise. When $a \notin (\lambda_1, \lambda_2]$, then $H^k(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) = 0$ for all k.*

Proof. We achieve a truncated long exact sequence from Proposition 10.20 and Proposition 9.3:

$$0 \longrightarrow H^0(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \xrightarrow{j_*^0} H^0(\Lambda; \mathscr{A}^{\boldsymbol{\psi}_*}) \xrightarrow{i_*^0} H^0(\Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \xrightarrow{\delta^1} H^1(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \longrightarrow 0.$$

Following the argument of Lemma 10.1, the map i_*^0 is injective and thus $H^0(\Lambda, \Lambda'; \mathscr{A}^{\boldsymbol{\psi}_*}) \cong 0$. Lemma 10.2 then yields:

$$H^1(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) \cong \frac{H^0(\Lambda'; \mathscr{A}^{\psi_*})}{\operatorname{im} i_*^0}$$

Note that $\operatorname{im} i_*^0 \cong H^0(\Lambda; \mathscr{A}^{\psi_*}) = \Gamma(\mathscr{A}^{\psi_*}) \cong \mathbb{Z}_2^3$. For $a \in (\lambda_1, \lambda_2]$, we have $H^0(\Lambda'; \mathscr{A}^{\psi_*}) = \Gamma(\mathscr{A}^{\psi_*}|_{\Lambda'}) \cong \mathbb{Z}_2^4$, which implies $H^1(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) \cong \mathbb{Z}_2$. Otherwise, $H^1(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) = 0$. □

Proposition 10.24. *Let $\Lambda' = (-\infty, a]$. If $a \in [\lambda_1, \lambda_2)$, then $H^k(\Lambda, \Lambda', \mathscr{A}^{\psi_*}) \cong \mathbb{Z}_2$ for $k = 1$, and is zero otherwise. When $a \notin [\lambda_1, \lambda_2)$, then $H^k(\Lambda, \Lambda'; \mathscr{A}^{\psi_*}) = 0$ for all k.*

Proof. An identical argument as in the proof of Proposition 10.23. □

Theorem 10.25. *Let ϕ_* be a parametrized dynamical system conjugate to the above parametrization ψ_* of the S-shaped bifurcation. Then,*

$$\mathscr{A}^{\phi_*} \text{ is acyclic and } H^0(\Lambda; \mathscr{A}^{\phi_*}) \cong \mathbb{Z}_2^3.$$

Moreover, there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$H^k(\Lambda, \Lambda'; \mathscr{A}^{\phi_*}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 1 \text{ and } a \in (\lambda_1, \lambda_2] \\ 0 & \text{otherwise,} \end{cases} \quad \text{with} \quad \Lambda' = [a, \infty),$$

$$H^k(\Lambda, \Lambda'; \mathscr{A}^{\phi_*}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 1 \text{ and } a \in [\lambda_1, \lambda_2) \\ 0 & \text{otherwise,} \end{cases} \quad \text{with} \quad \Lambda' = (-\infty, a].$$

Proof. Apply Theorem 8.7, Proposition 10.20, and Propositions 10.23 and 10.24. □

Remark 10.26. Note that the relative cohomologies $H^k(\Lambda, \Lambda'; \mathscr{A}^{\phi_*})$ are the same for the transcritical and S-shaped bifurcation.

Remark 10.27. If we consider the S-shaped bifurcation on an interval $X = I = [-c, c]$, $c \gg 1$, with time space $\mathbb{T} = \mathbb{R}^+$ and parameter space $\Lambda = [-\lambda_0, \lambda_0]$, with $\lambda_0 = -c + c^3 - \epsilon$, $0 < \epsilon \ll 1$ we obtain the following sheaf cohomology:

$$\mathscr{A}^{\phi_*} \text{ is acyclic and } H^0(\Lambda; \mathscr{A}^{\phi_*}) \cong \mathbb{Z}_2.$$

Moreover, there exists a value $\lambda_0 \in \mathbb{R}$ such that

$$H^k(\Lambda, \Lambda'; \mathscr{A}^{\phi_*}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 1 \text{ and } a \in (\lambda_1, \lambda_2] \\ 0 & \text{otherwise,} \end{cases} \quad \text{with} \quad \Lambda' = [a, \infty) \cap \Lambda,$$

$$H^k(\Lambda, \Lambda'; \mathscr{A}^{\phi_*}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 1 \text{ and } a \in [\lambda_1, \lambda_2) \\ 0 & \text{otherwise,} \end{cases} \quad \text{with} \quad \Lambda' = (-\infty, a] \cap \Lambda.$$

For free attractor sheaf we have:

$$H^k(\Lambda; \mathscr{Att}^{\psi_*}) \cong \begin{cases} \mathbb{Z}_2^2 & \text{if } k = 0 \\ \mathbb{Z}_2 & \text{if } k = 1 \\ 0 & \text{if } k \geqslant 2 \end{cases},$$

which is clearly not acyclic.

10.3. Comparing the attractor and free attractor sheaves

The abelian attractor sheaf, as shown in Theorem 10.25, is acyclic for the S-shaped bifurcation. Proposition 10.22 demonstrates nontrivial trivial cohomology in dimension one for the free attractor sheaf. Consider the following section in $\boldsymbol{\psi}_*^{-1}\mathscr{S}^{\mathsf{Att}}(\Lambda_1 \cap \Lambda_2)$:

$$s\colon \lambda \mapsto (\lambda, \boldsymbol{\psi}_*(\lambda), \{y_\lambda, x_\lambda\}).$$

$\ell(s)$ is not the restriction of any section from $\mathscr{A}^{\psi_*}(\Lambda_1)$ or $\mathscr{A}^{\psi_*}(\Lambda_2)$. Likewise, the corresponding generator for s in $\mathscr{Att}^{\psi_*}((\lambda_1, \lambda_2))$ is not the restriction of any section from $\mathscr{Att}^{\psi_*}(\Lambda_1)$ or $\mathscr{Att}^{\psi_*}(\Lambda_2)$. However, in $\mathscr{A}^{\psi_*}(\Lambda_1 \cap \Lambda_2)$:

$$\ell(s) = \rho_1(e_1') + \rho_2(e_1''),$$

where ρ_1, ρ_2, e_1', and e_1'' are defined in Proposition 10.20. Thus, s lies in the image of $\rho_1 - \rho_2$. The analogous statement cannot be said for s's corresponding generator in $\mathscr{Att}^{\psi_*}((\lambda_1, \lambda_2))$; it does not lie in the image of $\rho_{1,2}^2 - \rho_{1,2}^1$ as defined in 10.22. So while this generator does contribute a nontrivial cohomology class to $H^1(\Lambda, \mathscr{Att}^{\psi_*})$, $\ell(s)$ does not. This highlights the difference between the purely formal addition operation in $\mathscr{Att}^{\psi_*}(\Lambda_1 \cap \Lambda_2)$ and the addition operation in $\mathscr{A}^{\psi_*}(\Lambda_1 \cap \Lambda_2)$ constructed from the lattice structure of $\boldsymbol{\psi}_*^{-1}\mathscr{S}^{\mathsf{Att}}(\Lambda_1 \cap \Lambda_2)$.

Data availability

No data was used for the research described in the article.

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Appendix A. Table with important definitions

	Notation	Description	Reference
Dynamics	$\mathbb{T}$	Time space, either $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}$, or $\mathbb{R}_+$	Sect 2
	$\mathrm{Inv}_\phi(U)$	Maximal invariant set in U	Sect 3, pg 131
	$\omega_\phi(U)$	Omega limit set of U	Sect 3, pg 131
	$\alpha_\phi(U)$	Alpha limit set of U	Rmk 3.6

(*continued on next page*)

	Notation	Description	Reference
	$\mathsf{ANbhd}(\phi)$	Lattice of attracting neighborhoods for ϕ	Sect 3, pg 131
	$\mathsf{Att}(\phi)$	Lattice of attractors for ϕ	Sect 3, pg 131
	$\mathsf{Rep}(\phi)$	Lattice of repellers for ϕ	Rmk 3.6
	$\mathsf{Morse}(\phi)$	Meet semilattice of Morse sets for ϕ	Sect 3, pg 131
	$\mathsf{MRepr}(\phi)$	Lattice of Morse representations for ϕ	Sect 5, pg 143
	$\mathsf{C}_{\mathsf{Att}}(A, A')$	Conley form of two attractors	Sect 6, pg 145
Category Theory	$\mathbf{DS}(\mathbb{T}, X)$	Category of dynamical systems	Sect 2, pg 129
	$\mathsf{ob}(\mathbf{C}), \mathsf{hom}(\mathbf{C})$	Objects and morphisms of a category $\mathbf{C}$	[29]
	$\mathsf{hom}(\phi, \psi)$	Morphisms between two objects	[29]
	$\mathsf{F}\colon \mathbf{D} \to \mathbf{C}$	Universe functor	Sect 4.1, pg 133
	$\mathsf{F}_0 \in \mathbf{C}$	Value of universe functor	Sect 4.1, pg 133
	$\Pi[\mathsf{E}]$	Category of elements for a functor E	[29,30]
	$\Phi[\mathsf{E}; U]$	Objects ϕ for which $U \in \mathsf{E}(\phi)$	Sect 4.1, pg 134
	$\Theta[\mathsf{w}; U]$	Partial section functor on $\Phi[\mathsf{E}; U]$	Sect 4.1, pg 135
	$(\mathsf{G}, \mathsf{E}, \mathsf{w})$	Continuation frame	Definition 4.2
Order Theory	$\mathbf{BDLat}$	Category of bounded, distributive lattices	[6] Sect 2
	$\mathbf{MLat}$	Category of bounded, meet-semilattices	Rmk 5.3
	$\mathrm{sub}_{\mathrm{F}}\colon \mathbf{Lat} \to \mathbf{Lat}$	Lattice of finite sublattices functor	Sect 5.2, pg 141
	$\overleftarrow{U}$	Unique immediate predecessor of U	[6]
	$\mathsf{O}\colon \mathbf{Poset} \to \mathbf{BDLat}$	Down-set functor	[6]
	$\mathsf{J}\colon \mathbf{Lat} \to \mathbf{Poset}$	Poset of join-irreducibles of U	[6]
	$\mathsf{B}\colon \mathbf{BDLat} \to \mathbf{Bool}$	Booleanization functor	[6]
	$\mathsf{R}\colon \mathbf{BDLat} \to \mathbf{Ring}$	(Boolean) lattice ring of L	Sect 6.3, pg 147
	$\mathbb{Z}_2\colon \mathbf{BDLat} \to \mathbf{Ring}$	Lattice algebra of L	Sect 6.3, pg 148
Sheaf Theory	$\mathscr{S}^{\mathsf{G}}\colon \mathscr{O}(\mathbf{D}) \to \mathbf{Set}$	Sheaf of sections for $\Pi[\mathsf{G}]$	Sect 7.2
	$\Gamma(\mathscr{S}^{\mathsf{G}})$	Set of global sections for $\mathscr{S}^{\mathsf{G}}$	Sect 7.2
	$\mathscr{F}_\phi$	Stalk of a sheaf $\mathscr{F}$ at ϕ	Rmk 7.9, [44]
	$\mathscr{S}^{\mathsf{Att}}$	Attractor lattice sheaf	Sect 7.3, pg 153
	$\mathscr{A}^{\boldsymbol{\phi}_*}$	Attractor sheaf for $\boldsymbol{\phi}_*$	Sect 8, pg 159
	$\mathscr{A}\!\mathit{tt}^{\boldsymbol{\phi}_*}$	Free attractor sheaf for $\boldsymbol{\phi}_*$	Sect 8, pg 159
	$H^*(\Lambda; \mathscr{F})$	Sheaf cohomology of a sheaf $\mathscr{F}$ on Λ	[44]
	$H^*(\Lambda, \Lambda'; \mathscr{F})$	Relative sheaf cohomology	[44]

Appendix B. Functorial properties of attractors

Proof of Lemma 3.2. For $t \geqslant 0$, we have $\phi_t(h^{-1}(U)) \subset (h^{-1} \circ h \circ \phi_t)(h^{-1}(U))$. Since h is a quasiconjugacy, we have $(h^{-1} \circ h \circ \phi_t)(h^{-1}(U)) = h^{-1}(\psi_t^\dagger((h \circ h^{-1})(U))) \subset h^{-1}(\psi_t^\dagger(U))$ and thus

$$\phi_t(h^{-1}(U)) \subset h^{-1}(\psi_t^\dagger(U)), \quad \forall t \geqslant 0.$$

The inequality for ω now follows from elementary properties of inverse images and closures:

$$\begin{aligned}\omega_\phi(h^{-1}(U)) &= \bigcap_{t\geqslant 0} \mathrm{cl} \bigcup_{s\geqslant t} \phi_s\big(h^{-1}(U)\big) \subset \bigcap_{t\geqslant 0} \mathrm{cl} \bigcup_{s\geqslant t} h^{-1}\big(\psi_s^\dagger(U)\big) = \bigcap_{t\geqslant 0} \mathrm{cl}\, h^{-1}\Big(\bigcup_{s\geqslant t} \psi_s^\dagger(U)\Big)\\ &\subset \bigcap_{t\geqslant 0} h^{-1}\Big(\mathrm{cl} \bigcup_{s\geqslant t} \psi_s^\dagger(U)\Big) = h^{-1}\Big(\bigcap_{t\geqslant 0} \mathrm{cl} \bigcup_{s\geqslant t} \psi_s^\dagger(U)\Big)\end{aligned}$$

$$= h^{-1}\Big(\bigcap_{t\geqslant 0}\mathrm{cl}\bigcup_{x\in U}\bigcup_{\sigma\geqslant\tau(t,x)}\psi_\sigma(x)\Big) = h^{-1}\Big(\bigcap_{\tau\geqslant 0}\mathrm{cl}\bigcup_{\sigma\geqslant\tau}\psi_\sigma(U)\Big) = h^{-1}(\omega_\psi(U)),$$

which uses the invertibility of the parametrization function τ. Finally applying ω_ϕ we obtain

$$\omega_\phi(h^{-1}(U)) = \omega_\phi(\omega_\phi(h^{-1}(U))) \subset \omega_\phi(h^{-1}(\omega_\psi(U))) \subset \omega_\phi(h^{-1}(U))$$

so that

$$\omega_\phi(h^{-1}(U)) = \omega_\phi(h^{-1}(\omega_\psi(U))), \tag{B.1}$$

which completes the proof. □

Proof of Remark 3.6. To deal with negative times we define $\tau(-t,x) := \tau(t,x)$ in which case

$$\psi^\dagger_{-t} = \psi\big(\tau(-t,\cdot),\cdot\big) = \psi\big(-\tau(t,\cdot),\cdot\big) = \big(\psi^\dagger_t\big)^{-1}.$$

Let $x \in \phi_{-t}(h^{-1}(U))$ so that $\phi_t(x) \in h^{-1}(U)$. Then, by the quasiconjugacy condition $h(\phi_t(x)) = \psi^\dagger_t(h(x)) \in U$, and therefore $h(x) \in \psi^\dagger_{-t}(U)$. This yields $x \in h^{-1}(\psi^\dagger_{-t}(U))$. Summarizing we have

$$\phi_{-t}(h^{-1}(U)) \subset h^{-1}(\psi^\dagger_{-t}(U)), \quad \forall t \geqslant 0.$$

The remainder of the proof is similar to the proof of Lemma 3.2. □

Proof of Proposition 3.4. Since A is an attractor for ψ, there exists an attracting neighborhood U such that $\omega_\psi(U) = A$. By Eqn. (B.1) we have

$$\omega_\phi\big(h^{-1}(U)\big) = \omega_\phi\big(h^{-1}(\omega_\psi(U))\big) = \omega_\phi\big(h^{-1}(A)\big),$$

which proves that $\omega_\phi\big(h^{-1}(A)\big)$ is an attractor for ϕ, since we already know $h^{-1}(U)$ is an attracting neighborhood for ϕ.

Therefore, for a quasiconjugacy $\tau \times h \in \mathsf{hom}(\phi,\psi)$, the map $\omega_\phi \circ h^{-1}\colon \mathsf{Att}(\psi) \to \mathsf{Att}(\phi)$ is well defined. It remains to show that the latter is a lattice homomorphism. Preservation of joins is clear, cf. Property (v) for omega-limit sets. Let $A, A' \in \mathsf{Att}(\psi)$, then

$$\begin{aligned}\omega_\phi(h^{-1}(A \wedge A'))) &= \omega_\phi(h^{-1}(\omega_\psi(A \cap A'))) \subset \omega_\phi(h^{-1}(A \cap A')) = \omega_\phi\big(h^{-1}(A) \cap h^{-1}(A')\big)\\ &= \omega_\phi\big(\omega_\phi\big(h^{-1}(A) \cap h^{-1}(A')\big)\big) \subset \omega_\phi\big(\omega_\phi(h^{-1}(A)) \cap \omega_\phi(h^{-1}(A'))\big)\\ &= \omega_\phi(h^{-1}(A)) \wedge \omega_\phi(h^{-1}(A'))\end{aligned}$$

Idempotency of ω_ϕ and Equation (3) imply

$$\begin{aligned}\omega_\phi(h^{-1}(A)) \wedge \omega_\phi(h^{-1}(A')) &= \omega_\phi\big(\omega_\phi(h^{-1}(A)) \cap \omega_\phi(h^{-1}(A'))\big)\\ &\subset \omega_\phi\big(h^{-1}(\omega_\psi(A)) \cap h^{-1}(\omega_\psi(A'))\big) = \omega_\phi\big(h^{-1}(\omega_\psi(A) \cap \omega_\psi(A'))\big)\\ &= \omega_\phi(h^{-1}(A) \cap h^{-1}(A')) = \omega_\phi(h^{-1}(A \cap A'))\\ &= \omega_\phi\big(\omega_\phi(h^{-1}(A \cap A'))\big) \subset \omega_\phi(h^{-1}(\omega_\psi(A \cap A')))\\ &= \omega_\phi(h^{-1}(A \wedge A'))),\end{aligned}$$

which proves that

$$\omega_\phi\big(h^{-1}(A \wedge A'))\big) = \omega_\phi(h^{-1}(A)) \wedge \omega_\phi(h^{-1}(A')),$$

and thus $\omega_\phi \circ h^{-1}\colon \mathsf{Att}(\psi) \to \mathsf{Att}(\phi)$ is a lattice homomorphism. □

Proof of Remark 3.5. If $\tau \times h \in \mathsf{hom}(\phi, \psi)$ is a conjugacy, then

$$h\big(\phi_t(x)\big) = \psi_t^\dagger\big(h(x)\big). \tag{B.2}$$

Define $y = h(x)$ and $s = \tau\big(t, h^{-1}(y)\big)$. Since h is a homeomorphism, we obtain $\tau^{-1}(s, y)$, and therefore

$$\phi_s^\dagger\big(h^{-1}(y)\big) = h^{-1}\big(\psi_s(y)\big), \tag{B.3}$$

where $\phi_s^\dagger = \psi\big(\tau^{-1}(s, \cdot), \cdot\big)$. This proves that $\tau^{-1} \times h^{-1} \in \mathsf{hom}(\psi, \phi)$ is a conjugacy.

Let $A \in \mathsf{Att}(\psi)$, then by Proposition 3.4, we have $\omega_\phi\big(h^{-1}(A)\big) \in \mathsf{Att}(\phi)$. By Equation (B.3) we have $\phi_s^\dagger\big(h^{-1}(A)\big) = h^{-1}\big(\psi_s(A)\big) = h^{-1}(A)$ for all $s \geqslant 0$, which proves invariance of $h^{-1}(A)$. Furthermore, since h is a homeomorphism, it follows that $h^{-1}(A)$ is closed, and thus $\omega_\phi\big(h^{-1}(A)\big) = h^{-1}(A)$, which proves that $h^{-1}(A) \in \mathsf{Att}(\phi)$. Similarly, $h(A) \in \mathsf{Att}(\psi)$ for all $A \in \mathsf{Att}(\phi)$. □

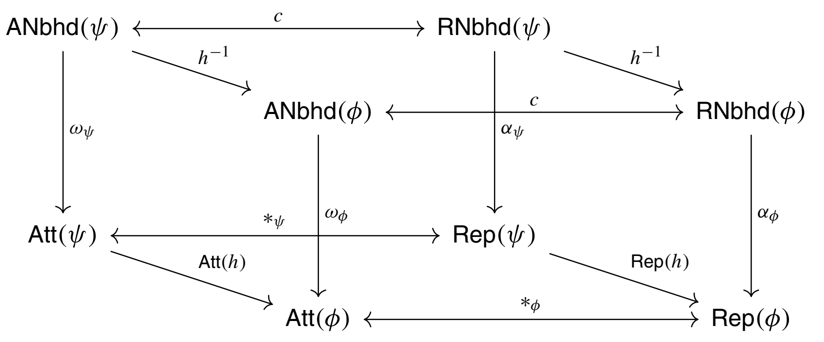
Appendix C. Repellers

In Remarks 3.6 and 5.5 we indicated that one can also the construct continuation frames $(\mathsf{Rep}, \mathsf{RNbhd}, \alpha)$ based repelling neighborhoods and repellers which yields the étalé space $\Pi[\mathsf{Rep}]$. For a dynamical system $\phi\colon \mathbb{T}^+ \times X \to X$ we define $\phi_{-t} := \phi_t^{-1}$ as the inverse image. The map $\phi(-t, x)$ also satisfies the semigroup property. This allows us to define the notion *alpha-limit set* as

$$\alpha_\phi(U) := \bigcap_{t \geqslant 0} \mathrm{cl} \bigcup_{s \geqslant t} \phi_{-s}(U).$$

Some properties of $\alpha_\phi(U)$ are: (i) $\alpha_\phi(U)$ is compact, closed, (ii) $\alpha_\phi(U)$ is a forward-backward invariant set for the dynamics, (iii) $\alpha_\phi\big(\alpha_\phi(U)\big) \supset \alpha_\phi(U)$, (iv) $\alpha_\phi(U \cup V) = \alpha_\phi(U) \cup \alpha_\phi(V)$. A neighborhood $U \subset X$ is called a *repelling neighborhood* if $\alpha_\phi(U) \subset \mathrm{int}\, U$. Repelling neighborhoods form a bounded, distributive lattice denoted by $\mathsf{RNbhd}(\phi)$. The binary operations are $\cap$ and $\cup$. A subset $A \subset X$ is called a *repeller* if there exists an repelling neighborhood $U \subset X$ such that $R = \alpha_\phi(U)$, which is a neighborhood of R by definition. Repellers are compact, closed, forward-backward invariant sets and compose a bounded, distributive lattice $\mathsf{Rep}(\phi)$ with binary operations $\cup$ and $\cap$. As before $\phi \mapsto \mathsf{RNbhd}(\phi)$ and $\phi \mapsto \mathsf{Rep}(\phi)$ define the contravariant functors RNbhd and Rep from $\mathbf{DS}(\mathbb{T}, X) \to \mathbf{BDLat}$. The functor $\mathsf{RNbhd}\colon \mathbf{DS}(\mathbb{T}, X) \to \mathbf{BDLat}$ is a stable structure and $(\mathsf{Rep}, \mathsf{RNbhd}, \alpha)$ forms a continuation frame in a similar way. From the continuation frame $(\mathsf{Rep}, \mathsf{RNbhd}, \alpha)$ we obtain the étalé space $(\Pi[\mathsf{Rep}], \pi)$.

For a dynamical system ϕ consider the duality isomorphism $A \mapsto A^*$, $A \in \mathsf{Att}(\phi)$. Since for $U \in \mathsf{ANbhd}(\phi)$ the maps $U \mapsto U^c$ and $\omega_\phi(U) \mapsto \alpha_\phi(U^c)$ define lattice isomorphisms we also have the natural transformations ${}^c\colon \mathsf{ANbhd} \Longleftrightarrow \mathsf{RNbhd}$ and ${}^*\colon \mathsf{Att} \Longleftrightarrow \mathsf{Rep}$. This yields the following commutative diagram:



where $\mathsf{Att}(h) = \omega_\phi \circ h^{-1}$ and $\mathsf{Rep}(f) := *_\phi \circ \omega_\phi \circ h^{-1} \circ *_\psi$. This asymmetry between attractors and repellers is typical for noninvertible systems. For invertible systems the symmetry is restored so that $\mathsf{Rep}(h) = \alpha_\phi \circ h^{-1}$.

Appendix D. Function spaces and the compact-open topology

We recall some basic facts about topologies on function spaces of continuous functions. Let X and Y be arbitrary topological spaces and let $C(X, Y)$ the denote the set of all continuous maps $f\colon X \to Y$. A topology on $C(X, Y)$ which is of particular importance is the *compact-open topology* which is defined a subbasis of sets of the form

$$O(K, U) := \big\{ f \mid f(K) \subset U \text{ for } K \text{ compact in } X \text{ and } U \text{ open in } Y \big\},$$

where K ranges over all compact subsets in X and U ranges over all open subsets in Y, cf. [51]. If X is a locally compact, Hausdorff space then the compact-open topology is the weakest topology such that the map $(f, x) \mapsto f(x)$, $f \in C(X, Y)$, is continuous, cf. [52, Cor. 1.2.4]. If X is compact and Y is a metric space with metric d, then the compact-open topology corresponds with the metric topology on $C(X, Y)$ given by the metric:

$$d(f, g) = \sup_{x \in X} d(f(x), g(x)), \qquad f, g \in C(X, Y),$$

cf. [53,54].

Let Λ be an arbitrary topological space. For a continuous map $h\colon \Lambda \times X \to Y$ we define the *transpose* of h by:

$$h_*\colon \Lambda \to C(X,Y), \qquad \lambda \mapsto h_*(\lambda) = h^\lambda := h(\lambda, \cdot).$$

Following the terminology in [55] we say that a topology on $C(X,Y)$ is *weak* if continuity of h implies continuity of the transpose h_*, and a topology is *strong* if continuity of the transpose h_* implies continuity of h. For arbitrary topological spaces X, Y and Λ the compact-open topology is a weak topology on $C(X,Y)$, i.e. h continuous implies that h_* is continuous, cf. [51, Lemma 1], [53]. If X is regular and locally compact (in particular for locally compact, Hausdorff spaces), then the compact-open topology is both weak and strong, i.e. h is continuous if and only if h_* is continuous, cf. [51, Theorem 1], [53]. This implies that for regular and locally compact spaces X the compact-open topology on $C(X,Y)$ is both weak and strong, which is also referred to as an *exponential topology*, cf. [55]. The latter is unique. Finally, the map $h \mapsto h_*$ is an embedding when both Λ and X are Hausdorff spaces. The map is a homeomorphism when Λ is Hausdorff and X is locally compact, Hausdorff, cf. [53].

For a compact metric space (X,d) define $\big(\mathsf{H}(X), d_\mathsf{H}\big)$ to be the metric space of compact subsets of X equipped with the Hausdorff metric d_H. Every continuous function $f\colon X \to Y$ induces a continuous function $f^\mathsf{H}\colon \mathsf{H}(X) \to \mathsf{H}(Y)$, which sends compact subsets to their image under the function f. Recall the Hausdorff metric:

$$d_\mathsf{H}(K,K') := \max\left\{\sup_{x\in K}\inf_{x'\in K'} d(x,x'), \sup_{x'\in K'}\inf_{x\in K} d(x,x')\right\}, \quad K, K' \in \mathsf{H}(X).$$

Lemma D.1. *Let X, Y be compact metric spaces, Λ a topological space, and $h\colon \Lambda \times X \to Y$ continuous map. Then, the function*

$$h^\mathsf{H}\colon \Lambda \times \mathsf{H}(X) \to \mathsf{H}(Y) \qquad (\lambda, K) \mapsto h^\mathsf{H}\big(\{\lambda\}\times K\big)$$

is continuous.

Proof. We will first prove the assignment

$$\mathsf{D}\colon C(X,Y) \to C(\mathsf{H}(X), \mathsf{H}(Y)) \qquad f \mapsto \mathsf{D}(f) := f^\mathsf{H},$$

is continuous. Let $f, g \in C(X,Y)$. Then, $d_C(f,g) = \sup_{x\in X} d(f(x), g(x))$ and

$$d_{C_\mathsf{H}}(f^\mathsf{H}, g^\mathsf{H}) = \sup_{K\in\mathsf{H}(X)} d_\mathsf{H}\big(f(K), g(K)\big).$$

Since $d(y,y') \leqslant \sup_{x\in X} d(f(x), g(x)) = d(f,g)$ for any choice of $y \in f(K)$, $y' \in g(K)$ it follows that $d_{C_\mathsf{H}}(f^\mathsf{H}, g^\mathsf{H}) \leqslant d_C(f,g)$. Moreover, since points are compact subsets, the reversed inequality holds as well:

$$d_C(f,g) = \sup_{x\in X} d_\mathsf{H}\big(f(\{x\}), g(\{x\})\big) \leqslant d_{C_\mathsf{H}}(f^\mathsf{H}, g^\mathsf{H}),$$

which proves that D is an isometry implying its continuity. The metric topology on $C(\mathsf{H}(X), \mathsf{H}(Y))$ coincides with the compact-open topology and therefore h_H is continuous if and only if its *transpose* h_*^h, given by

$$h_*^{\mathsf{H}}\colon \Lambda \to C\big(\mathsf{H}(X), \mathsf{H}(Y)\big), \qquad \lambda \mapsto h_*^{\mathsf{H}}(\lambda) = h_{\mathsf{H}}^{\lambda} := h_{\mathsf{H}}\big(\{\lambda\}, \cdot\big),$$

is continuous, [51,55]. Note that $h_*^{\mathsf{H}} = \mathsf{D} \circ h_*$ which proves that h_*^{H} is continuous, which completes the proof. □

Remark D.2. In this paper we abuse notation by writing $h(K)$, $K \in \mathsf{H}(X)$ denoting $h^{\mathsf{H}}(K)$ in accordance with the analogous notation for $h(U) = \{y = h(x) \mid x \in U\}$, $U \subset X$.

Appendix E. Sheaf cohomology

Let us recall the most important principles of sheaf cohomology. Let $\mathscr{F}\colon \mathscr{O}(\mathbf{D}) \to \mathbf{Ab}$ be a sheaf of abelian groups over a topological space $\mathbf{D}$, where $\mathscr{O}(\mathbf{D})$ is the posetal category of open sets in $\mathbf{D}$. Here we review the construction of the Godement resolution for computing sheaf cohomology. A more detailed description can be found in [44]. Consider the following presheaf on $\mathbf{D}$:

$$\mathscr{C}^0(\mathbf{D}; \mathscr{F})\colon \mathscr{O}(\mathbf{D}) \to \mathbf{Ab}, \qquad \Omega \mapsto \prod_{\phi \in \Omega} \mathscr{F}_\phi,$$

where $\mathscr{F}_\phi$ is the stalk of $\mathscr{F}$ at ϕ with the restriction maps being the canonical projections. It holds that $\mathscr{C}^0(\mathbf{D}; \mathscr{F})$ is again a sheaf. There is a corresponding injection of sheaves $\epsilon^0\colon \mathscr{F} \to \mathscr{C}^0(\mathbf{D}; \mathscr{F})$, which sends sections on an open set $\Omega \subset \mathbf{D}$ to the product of their germs in $\prod_{\phi\in\Omega} \mathscr{F}_\phi$. We let $\mathscr{K}^1(\mathbf{D}; \mathscr{F})$ be the cokernel sheaf of this mapping, accompanied by the quotient map $\partial^0\colon \mathscr{C}^0(\mathbf{D}; \mathscr{F}) \to \mathscr{K}^1(\mathbf{D}; \mathscr{F})$. This gives us an exact sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\epsilon^0} \mathscr{C}^0(\mathbf{D}; \mathscr{F}) \xrightarrow{\partial^0} \mathscr{K}^1(\mathbf{D}; \mathscr{F}) \longrightarrow 0.$$

Define inductively

$$\mathscr{C}^n(\mathbf{D}; \mathscr{F}) = \mathscr{C}^0(\mathbf{D}; \mathscr{K}^n(\mathbf{D}; \mathscr{F})),$$
$$\mathscr{K}^{n+1}(\mathbf{D}; \mathscr{F}) = \mathscr{K}^1(\mathbf{D}; \mathscr{K}^n(\mathbf{D}; \mathscr{F})).$$

Then, by letting $d^n = \epsilon^{n+1} \circ \partial^n$, we obtain a long exact sequence:

$$0 \longrightarrow \mathscr{F} \xrightarrow{\epsilon^0} \mathscr{C}^0(\mathbf{D}; \mathscr{F}) \xrightarrow{d^0} \mathscr{C}^1(\mathbf{D}; \mathscr{F}) \xrightarrow{d^1} \mathscr{C}^2(\mathbf{D}; \mathscr{F}) \xrightarrow{d^2} \ldots$$

which is called the Godement resolution for $\mathscr{F}$. The functor $\Gamma\colon \mathbf{Sh}_{\mathbf{Ab}}(\mathbf{D}) \to \mathbf{Ab}$ which assigns to $\mathscr{F}$ the group of global sections $\mathscr{F}(\mathbf{D})$ on $\mathbf{D}$ is left-exact and yields the cochain complex

$$0 \longrightarrow \Gamma\big(\mathscr{F}\big) \xrightarrow{\epsilon^0} \Gamma\big(\mathscr{C}^0(\mathbf{D}; \mathscr{F})\big) \xrightarrow{d^0} \Gamma\big(\mathscr{C}^1(\mathbf{D}; \mathscr{F})\big) \xrightarrow{d^1} \Gamma\big(\mathscr{C}^2(\mathbf{D}; \mathscr{F})\big) \xrightarrow{d^2} \ldots$$

which we denote by $\big(C^k(\mathbf{D}; \mathscr{F}), d^k\big)$, where $C^k(\mathbf{D}; \mathscr{F}) := \Gamma\big(\mathscr{C}^k(\mathbf{D}; \mathscr{F})\big)$. The cohomology groups of the above cochain complex comprise the *sheaf cohomology*

$$H^k(\mathbf{D};\mathscr{F}) := H^k\big(C^*(\mathbf{D};\mathscr{F}), d^*\big).$$

For any subset $\Omega \subset \mathbf{D}$ we define $H^k(\Omega;\mathscr{F}) := H^k\big(\Omega;\mathscr{F}|_\Omega\big)$. In [44, Sect. I.6] 'family of supports' are used. Here we choose the family of supports to be all closed subsets of $\mathbf{D}$. The *support* of a global section $\sigma \in \mathscr{F}(\mathbf{D})$ is defined as $|\sigma| = \big\{\phi \in \mathbf{D} \mid \sigma(\phi) \neq 0\big\}$, where $\sigma(\phi)$ is defined as the image of σ in the stalk $\mathscr{F}_\phi$. Observe that $\sigma(\phi) = 0$ implies $\sigma\big|_\Omega = 0$ in some term $\mathscr{F}(\Omega)$ of the colimit $\mathscr{F}_\phi = \varinjlim_{\Omega\ni\phi}\mathscr{F}(\Omega)$. This demonstrates that the set upon which a section σ is zero is open. Its support is hence a closed subset and an element of the family of supports.

Let $\Xi \subset \mathbf{D}$ be a closed subset. Following [46, Exer. III.2.3] we define the functor Γ_Ξ: $\mathbf{Sh_{Ab}}(\mathbf{D}) \to \mathbf{Ab}$ which assigns to $\mathscr{F}$ the group of sections σ with support in Ξ. Using the above resolution for $\mathscr{F}$ we obtain the *sheaf cohomology with support in* Ξ which is denoted by

$$H^k_\Xi(\mathbf{D};\mathscr{F}) := H^k\Big(\Gamma_\Xi\big(\mathscr{C}^*(\mathbf{D};\mathscr{F})\big), d^*_\Xi\Big).$$

The latter is also referred to as *local sheaf cohomology*. For local sheaf cohomology we have the following long exact sequence

$$0 \longrightarrow H^0_\Xi(\mathbf{D};\mathscr{F}) \longrightarrow H^0(\mathbf{D};\mathscr{F}) \longrightarrow H^0(\mathbf{D}\smallsetminus\Xi;\mathscr{F}) \longrightarrow H^1_\Xi(\mathbf{D};\mathscr{F}) \longrightarrow \cdots$$

In [44, Sect. II.12] yet another variation on sheaf cohomology is defined by considering relative sections in a sheaf. For a subset $\Omega \subset \mathbf{D}$ the embedding $i\colon \Omega \hookrightarrow \mathbf{D}$ implies the homomorphisms of sheaves $i^k\colon \mathscr{C}^k(\mathbf{D};\mathscr{F}) \to i\mathscr{C}^k(\Omega;\mathscr{F}|_\Omega)$. Define $\mathscr{C}^k(\mathbf{D},\Omega;\mathscr{F}) := \ker i^k$ which implies the following resolution

$$0 \longrightarrow \mathscr{F} \xrightarrow{\epsilon^0} \mathscr{C}^0(\mathbf{D},\Omega;\mathscr{F}) \xrightarrow{d^0} \mathscr{C}^1(\mathbf{D},\Omega;\mathscr{F}) \xrightarrow{d^1} \mathscr{C}^2(\mathbf{D},\Omega;\mathscr{F}) \xrightarrow{d^2} \ldots$$

The latter induces *relative sheaf cohomology*

$$H^k(\mathbf{D},\Omega;\mathscr{F}) := H^k\Big(\Gamma\big(\mathscr{C}^*(\mathbf{D},\Omega;\mathscr{F})\big), d^*_{\mathbf{D},\Omega}\Big).$$

As before we have the following long exact sequence

$$0 \longrightarrow H^0(\mathbf{D},\Omega;\mathscr{F}) \longrightarrow H^0(\mathbf{D};\mathscr{F}) \longrightarrow H^0(\Omega;\mathscr{F}) \longrightarrow H^1(\mathbf{D},\Omega;\mathscr{F}) \longrightarrow \cdots$$

cf. [44, Sect. II.12]. There is a relation between local and relative sheaf cohomology. For a closed subset $\Xi \subset \mathbf{D}$ we have

$$H^k_\Xi(\mathbf{D};\mathscr{F}) \cong H^k(\mathbf{D},\mathbf{D}\smallsetminus\Xi;\mathscr{F}).$$

A number of properties of relative sheaf cohomology can be summarized as follows:

(i) For a triple $\Omega'' \subset \Omega' \subset \Omega$ we have the long exact sequence

$$0 \longrightarrow H^0(\Omega,\Omega';\mathscr{F}) \longrightarrow H^0(\Omega,\Omega'';\mathscr{F}) \longrightarrow H^0(\Omega',\Omega'';\mathscr{F}) \longrightarrow H^1(\Omega,\Omega';\mathscr{F}) \longrightarrow \cdots$$

(ii) For any $\mathrm{cl}\,\Omega' \subset \mathrm{int}\,\Omega$ we have

$$H^k(\mathbf{D}, \Omega; \mathscr{F}) \cong H^k(\mathbf{D} \smallsetminus \Omega', \Omega \smallsetminus \Omega'; \mathscr{F});$$

(iii) For an exact sequence of sheaves $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ we have the long exact sequence

$$0 \longrightarrow H^0(\mathbf{D}, \Omega; \mathscr{F}') \longrightarrow H^0(\mathbf{D}, \Omega; \mathscr{F}) \longrightarrow H^0(\mathbf{D}, \Omega; \mathscr{F}'') \longrightarrow H^1(\mathbf{D}, \Omega; \mathscr{F}) \longrightarrow \cdots$$

Sheaf cohomology may be hard to compute in concrete situations. The Čech cohomology construction provides a good approach to sheaf cohomology and is isomorphic to sheaf cohomology when $\mathbf{D}$ is a paracompact, Hausdorff topological space. For the Čech construction we use coverings of $\mathbf{D}$.

Let $\mathscr{W} = \{\Omega_i\}_{i\in I}$ be a open covering for $\mathbf{D}$ and denote intersections of elements in $\mathscr{W}$ by $\Omega_{i_0\ldots i_n} = \bigcap_{k=0}^n \Omega_{i_k}$. The notation $\Omega_{i_0\ldots\widehat{i_m}\ldots i_n} := \Omega_{i_0\ldots i_{m-1}i_{m+1}\ldots i_n}$ omits Ω_{i_m} from the intersection. Note that $\Omega_{i_0\ldots i_n} \subset \Omega_{i_0\ldots\widehat{i_m}\ldots i_n}$ and

$$\rho^m_{i_0\ldots i_n} := \mathscr{F}(\iota)\colon \mathscr{F}(\Omega_{i_0\ldots\widehat{i_m}\ldots i_n}) \to \mathscr{F}(\Omega_{i_0\ldots i_n})$$

is the associated restriction map. Define the Čech cochain groups

$$C^k(\mathscr{W}; \mathscr{F}) := \prod_{(i_0\ldots i_k)\in I^{k+1}} \mathscr{F}(\Omega_{i_0\ldots i_k}),$$

and associated Čech coboundary operators

$$\delta^k_{\mathscr{F}}\colon C^k(\mathscr{W}; \mathscr{F}) \to C^{k+1}(\mathscr{W}; \mathscr{F}),$$

given by

$$\delta^k_{\mathscr{F}}(\sigma)_{i_0\ldots i_{k+1}} = \sum_{j=0}^{k+1} (-1)^j \rho^j_{i_0\ldots i_k}(\sigma_{i_0,\ldots\widehat{i_j}\ldots i_{k+1}}).$$

This defines the *Čech cohomology of* $\mathscr{W}$ is defined as $H^k(\mathscr{W}; \mathscr{F}) := H^k\Big(C^*(\mathscr{W}; \mathscr{F}), \delta^*_{\mathscr{F}}\Big)$. If one has a total ordering on I, one can define the *ordered Čech complex*

$$\overline{C}^k(\mathscr{W}; \mathscr{F}) := \prod_{i_0<\ldots<i_k} \mathscr{F}(\Omega_{i_0\ldots i_k}),$$

with the same coboundary operator. The cohomology of this complex is isomorphic to that of the standard Čech cohomology. The usual construction of Čech cohomology then yields *Čech sheaf cohomology of* $\mathbf{D}$ as

$$\check{H}^k(\mathbf{D}; \mathscr{F}) := \varinjlim_{\mathscr{W}} H^k(\mathscr{W}; \mathscr{F}).$$

There exists a natural homomorphism $H^k(\mathbf{D};\mathscr{F}) \to \check{H}^k(\mathbf{D};\mathscr{F})$, which is an isomorphism for $k \leqslant 1$ and for all k is $\mathbf{D}$ is a paracompact, Hausdorff topological space. For sufficiently nice covers of $\mathbf{D}$, Leray's Theorem yields immediate convergence of the limit.

Theorem D.3 *(cf. [44], Thm. III.4.13). Let $\mathscr{F}$ be a sheaf on a and $\mathscr{W}$ an open covering of* $\mathbf{D}$ *such that $\mathscr{F}\big|_{\Omega_{s^n}}$ is acyclic for all $s^n \in N(\mathscr{W})$, the nerve of $\mathscr{W}$. Then there is a canonical isomorphism*

$$H^*(\mathbf{D};\mathscr{F}) \cong \check{H}^*(\mathscr{W};\mathscr{F}).$$

The Čech construction can also be used to define relative Čech sheaf cohomology, cf. [56,57]. For an open subset $\Omega \subset \mathbf{D}$. A covering $\mathscr{W}$ for $\mathbf{D}$ induces a covering $\mathscr{W}' = \{\Omega_i\}_{i\in I'}$ for Ω from sets $\Omega_i \cap \Omega \neq \emptyset$, $i \in I'$. The pair $(\mathscr{W},\mathscr{W}')$ is a covering of $(\mathbf{D},\Omega)$. The relative chain groups are defined as

$$C^k(\mathscr{W},\mathscr{W}';\mathscr{F}) := \Big\{\sigma \in C^k(\mathscr{W};\mathscr{F}) \mid \sigma_{i_0\cdots i_k} = 0 \quad \text{if} \quad i_0,\cdots,i_k \in I'\Big\}.$$

Via restriction we obtain the coboundary operator $\delta^k_{\mathscr{F}}\colon C^k(\mathscr{W},\mathscr{W}';\mathscr{F}) \to C^{k+1}(\mathscr{W},\mathscr{W}',\mathscr{F})$ and the associated relative Čech sheaf cohomologies $H^k(\mathscr{W},\mathscr{W}';\mathscr{F})$ and $\check{H}^k(\mathbf{D},\Omega;\mathscr{F})$.

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