The Algebra of Semi-flows: A Tale of Two Topologies

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Abstract

To capture the global structure of a dynamical system we reformulate dynamics in terms of appropriately constructed topologies, which we call *flow topologies*; we call this process *topologization*. This yields a description of a semi-flow in terms of a *bi-topological space*, with the first topology corresponding to the (phase) space and the second to the flow topology. A study of topology is facilitated through *discretization*, i.e. defining and examining appropriate finite sub-structures. Topologizing the dynamics provides an elegant solution to their discretization by discretizing the associated flow topologies. We introduce *Morse pre-orders*, an instance of a more general bi-topological discretization, which synthesize the space and flow topologies, and encode the directionality of dynamics. We describe how Morse pre-orders can be augmented with appropriate (co)homological information in order to describe invariance of the dynamics; this ensemble provides an *algebraization* of the semi-flow. An illustration of the main ingredients is provided by an application to the theory of discrete parabolic flows. Algebraization yields a new invariant for positive braids in terms of a bi-graded differential module which contains Morse theoretic information of parabolic flows.

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CHAPTER 1

Prelude

We introduce the point of view that dynamics can be studied as a topology. Thus an analysis of a dynamical system results in an analysis of two topologies (i.e. a bi-toplogical space): the first topology corresponds to the (phase) space, and the second to the dynamics. Topology and associated algebraic invariants have long played a prolific role in the theory of dynamical systems [12, 53, 56, 61]. Loosely stated, a dynamical system engenders topological data, both local (e.g. fixed points) and global (e.g. attractors) and the directionality of the dynamics organizes the data. The topological data have associated algebraic invariants which may further codify the relationship between local and global and often recover the invariance of the dynamics, i.e. provide information about the existence and structure of the invariant sets.

1.1. Topologization and discretization

The novelty of our approach – and the first theme of this text – is to formalize the dynamical system itself as a topology, and capture both topology and dynamics in the formalism of bi-topological spaces, i.e. a *topologization* of the dynamical system. Recall that a *semi-flow* on a topological space $(X, \mathscr{T})$[1] is a continuous map $\varphi\colon \mathbb{R}^+ \times X \to X$ such that

(i) $\varphi(0, x) = x$ for all $x \in X$, and
(ii) $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$ for all $s, t \in \mathbb{R}^+$ and $x \in X$.

For a semi-flow φ the backward image is denoted by $\varphi(-t, x)$, $t > 0$ and is defined as $\varphi(-t, x) := \{x' \in X \mid \varphi(t, x) = x'\}$. One way to regard a topological space is via a closure operator on the algebra of subsets of X, cf. Sect. 2.1. Using this point of view, a map φ as defined above yields a natural closure operator on X through backward or forward images. If we disregard the continuity of φ in $(X, \mathscr{T})$ this defines Alexandrov topologies on X which are denoted by $\mathscr{T}^-$ and $\mathscr{T}^+$ respectively, cf. Sect. 3.1. The topologies $\mathscr{T}^-$ and $\mathscr{T}^+$ record directionality of the flow, but discard the sense of time and invariance. They are also independent of the continuity properties of φ. We therefore define a topology which allows one to incorporate the continuity of φ in $(X, \mathscr{T})$ and which is more suited for capturing the important characteristics of dynamics such as invariance. We will refer to this topology as the

[1] A topology on X is denoted by $\mathscr{T}$.

(derived) *block-flow topology* denoted by $\mathscr{T}_{\blacksquare}^{-}$, cf. Sect. 3.1.2.[2] The topologies $\mathscr{T}$ and $\mathscr{T}_{\blacksquare}^{-}$ comprise the bi-topological space $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$ which becomes our model for a semi-flow φ.[3] Sets that are closed in $(X, \mathscr{T})$ and open $(X, \mathscr{T}_{\blacksquare}^{-})$, so called *pairwise clopen sets*, are closed attracting blocks for φ, cf. Thm. 3.8. Of course the block-flow topology $\mathscr{T}_{\blacksquare}^{-}$ discards some information about φ. However, we can define suitable (co)homology theories on $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$ which allow one to describe fundamental invariant structures of the dynamics, cf. Sect. 4.

The second theme of this paper concerns discretization of both topology and dynamics. The last few decades have seen ever more sophisticated uses of discrete approximation in order to explore global dynamical features [3, 10, 13, 39, 40, 41, 52]; these techniques are largely based on Conley theory, a topological generalization of Morse theory [12]. As before, we describe topology in terms of a closure algebra, which provides a powerful formalism for discretization and extends these techniques. Moreover, as we encode a semi-flow φ as a topology, topological discretization also provides a means of discretizing φ. If we describe a topological space via a closure algebra $\big(\mathsf{Set}(X), \mathrm{cl}\big)$, then discretization may be regarded as determining a finite sub-algebra in the category of closure algebras, cf. Sect. 2.1. A finite closure algebra may be represented by $\big(\mathsf{Set}(\mathcal{X}), \mathrm{cl}\big)$, where $\mathcal{X}$ is a finite set, and is equivalently described by a finite pre-order $(\mathcal{X}, \leqslant)$, called the *specialization pre-order* of the associated Alexandrov topology on $\mathcal{X}$. Duality of the latter pre-order defines a continuous map

$$\mathrm{disc}\colon X \twoheadrightarrow \mathcal{X},$$

which is called the *discretization map* and provides a discretization of $(X, \mathscr{T})$ by a finite topological space $(\mathcal{X}, \leqslant)$, cf. Eqn. (2.12). The elements of $\mathcal{X}$ are denoted by

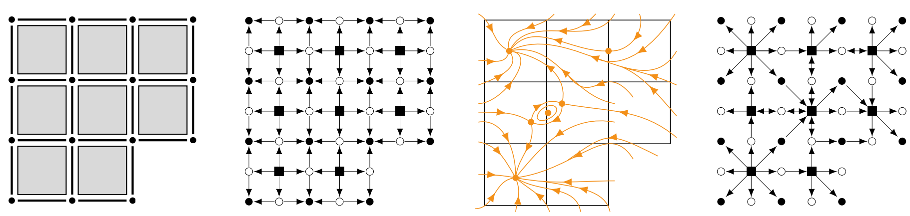

FIGURE 1.1. A discretization of $(X, \mathscr{T})$ with the associated face partial order $\leqslant$ [left 1 and 2]. Example of a semi-flow φ on X and pre-order $\leqslant_{\blacksquare}^{-}$ which is a discretization of the block-flow topology $\mathscr{T}_{\blacksquare}^{-}$ [right 3 and 4].

ξ and are called *cells* in $\mathcal{X}$. The closure algebra for a bi-topological space such as $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$ is given by $\big(\mathsf{Set}(X), \mathrm{cl}, \mathrm{cl}_{\blacksquare}^{-}\big)$ and by considering finite sub-structures

[2]As we have cast dynamics as topology, it is worthwhile to ask the question: what can dynamical systems theory say about topology? Section 4.2 may be regarded as steps in this direction.

[3]The topologies $\mathscr{T}$ and $\mathscr{T}_{\blacksquare}^{-}$ are in general related while the topologies $\mathscr{T}$ and $\mathscr{T}^{-}, \mathscr{T}^{+}$ are independently defined. The block-flow topology is not Alexandrov in general. An explanation of the notation of the various flow topologies we use is given in Sect.'s 3.1.2 and 6.1.1.

we obtain discretization maps $\mathrm{disc}\colon X \to \mathfrak{X}$ that are continuous with respect to two finite topologies $(\mathfrak{X}, \leqslant)$ and $(\mathfrak{X}, \leqslant_{\blacksquare}^{-})$. That is, for the bi-topological space $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$ a discretization is a finite bi-topological space $(\mathfrak{X}, \leqslant, \leqslant_{\blacksquare}^{-})$ with continuous discretization map $\mathrm{disc}\colon (X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-}) \twoheadrightarrow (\mathfrak{X}, \leqslant, \leqslant_{\blacksquare}^{-})$; this procedure is described for general bi-topological spaces in Sect. 2.5. Fig. 1.1 illustrates discretization of $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$ via two compatible pre-orders on a finite topological space $\mathfrak{X}$.

Attracting blocks, cf. Eqn. (3.3), play a central role in the study of the gradient-like and recurrent dynamics of a semi-flow φ. As earlier noted, the pairwise clopen sets in $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$ comprise the closed attracting blocks for φ. We can describe such sets in terms of a discretization which synthesizes both topologies. A *Morse pre-order*, cf. Defn. 3.18, is a pre-order $(\mathfrak{X}, \leqslant^{\dagger})$ such that both discretization maps $\mathrm{disc}\colon (X, \mathscr{T}) \twoheadrightarrow (\mathfrak{X}, \leqslant^{\dagger})$ and $\mathrm{disc}\colon (X, \mathscr{T}_{\blacksquare}^{-}) \twoheadrightarrow (\mathfrak{X}, \geqslant^{\dagger})$ are continuous; Morse pre-orders are particular instances of *antagonistic pre-orders*, which are defined purely in terms of bi-topological spaces, cf. Defn. 2.18. Closed sets in $(\mathfrak{X}, \leqslant^{\dagger})$ correspond to closed attracting blocks for φ, cf. Fig. 1.2[right].

The lattice $\mathsf{O}(\mathfrak{X}, \leqslant^{\dagger})$ of closed sets in $(\mathfrak{X}, \leqslant^{\dagger})$ can be represented by the down-sets of a finite poset $(\mathsf{SC}, \leqslant)$, cf. Eqn. (2.18), Fig. 1.2[left]. The map $\mathrm{dyn}\colon (\mathfrak{X}, \leqslant^{\dagger}) \twoheadrightarrow (\mathsf{SC}, \leqslant)$ is defined as the dual of $\mathsf{O}(\mathfrak{X}, \leqslant^{\dagger}) \cong \mathsf{O}(\mathsf{SC}, \leqslant) \rightarrowtail \mathsf{Set}(\mathfrak{X})$, cf. Eqn.'s (2.15)-(2.17).[4] Depending on the topology on $\mathfrak{X}$ the map dyn is order-preserving, or order-reversing[5] as is displayed in the following diagram:

$$
\begin{tikzcd}
 & (\mathfrak{X}, \leqslant) \arrow[d, "\mathrm{id}"] \arrow[dr, "\mathrm{dyn}"] & \\
(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-}) \arrow[ur, "\mathrm{disc}"] \arrow[r, bend left, "\mathrm{disc}"] \arrow[r, dashed, bend right, "\mathrm{disc}"'] \arrow[dr, "\mathrm{disc}"'] & (\mathfrak{X}, \leqslant^{\dagger}) \arrow[r, "\mathrm{dyn}"] & (\mathsf{SC}, \leqslant) \\
 & (\mathfrak{X}, \leqslant_{\blacksquare}^{-}) \arrow[u, dashed, "\mathrm{id}"] \arrow[ur, dashed, "\mathrm{dyn}"'] &
\end{tikzcd}
\tag{1.1}
$$

The composition $X \xrightarrow{\mathrm{disc}} \mathfrak{X} \xrightarrow{\mathrm{dyn}} \mathsf{SC}$ yields the T_0-discretization $\mathrm{tile}\colon X \to \mathsf{SC}$, cf. App. C.1; in particular, $\mathrm{tile}\colon (X, \mathscr{T}) \to (\mathsf{SC}, \leqslant)$ and $\mathrm{tile}\colon (X, \mathscr{T}_{\blacksquare}^{-}) \to (\mathsf{SC}, \geqslant)$ are continuous. The discretization tile defines a *Morse tessellation* with locally closed tiles $T = \mathrm{tile}^{-1}\mathcal{S}$, $\mathcal{S} \in \mathsf{SC}$, i.e. the sets T form a tessellation of X such that $\downarrow T$ is closed and $\varphi(t, x) \in \operatorname{int} \downarrow T$ for every $x \in T$ and for all tiles T, cf. Defn. 3.21 and [44, Defn. 8]. Fig. 1.2[right] illustrates how a Morse tessellation is obtained from a Morse pre-order. Conversely, if $(\mathsf{T}, \leqslant)$ is a Morse tessellation we obtain a Morse pre-order by defining $(\mathsf{T}, \leqslant)$ to be a Morse pre-order, and thus a discretization of $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$, cf. Sect. 3.3.1. For Boolean discretizations, i.e. discretizations for

[4]This map is christened dyn as it may be regarded as a grading by the dynamics. Combining (2.16) and (2.18) provides a formula for dyn, q.v. Thm. 3.29 and Eqn. (3.19).

[5]The dashed arrows in Figure 1.1 indicated order-reversing maps. We use this notation throughout the paper.

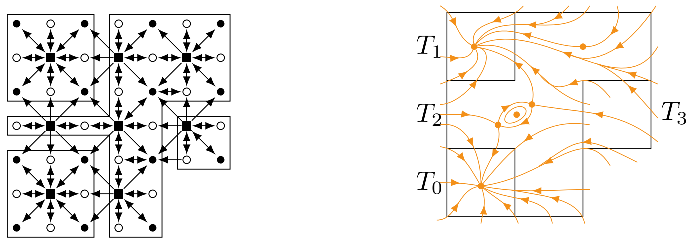

FIGURE 1.2. The antagonistic coarsening of $\leqslant$ and $\leqslant_{\blacksquare}^{-}$ in Fig. 1.1 makes a Morse pre-order $\leqslant^{\dagger}$ [left], with SC depicted as the outlined sets. The realization yields a Morse tessellation $(\mathsf{T}, \leqslant)$ [right].

which the closure operators for $(X, \mathscr{T})$ and $(\mathfrak{X}, \leqslant)$ commute,[6] cf. Defn. 2.9, we can exploit the fact that sets in $\mathsf{O}(\mathfrak{X}, \leqslant^{\dagger})$ correspond to regular closed sets, cf. Thm. 3.23. In this case we can define a pre-order $\leqslant^{\top}$ on the top cells[7] $\xi^{\top} \in \mathfrak{X}^{\top}$ such that $\mathsf{O}(\mathfrak{X}^{\top}, \leqslant^{\top}) \cong \mathsf{O}(\mathfrak{X}, \leqslant^{\dagger})$. Such a pre-order $(\mathfrak{X}^{\top}, \leqslant^{\top})$ is called a *condensed Morse pre-order* and drastically reduces the amount of data to analyze, cf. Sect.'s 3.4 and 5.4.1.[8] Condensed Morse pre-order do not lose information about the initial Morse pre-order and associated Morse tessellation. The map $\mathcal{U}^{\top} \mapsto \mathrm{cl}\, \mathcal{U}^{\top}$, with $\mathcal{U}^{\top} \in \mathsf{O}(\mathfrak{X}^{\top}, \leqslant^{\top})$, defines an injective lattice homomorphism $\mathrm{cl}\colon \mathsf{O}(\mathfrak{X}^{\top}, \leqslant^{\top}) \rightarrowtail \mathsf{O}(\mathfrak{X}, \leqslant)$, whose image $\mathsf{O}(\mathfrak{X}, \leqslant^{\dagger})$ is the lattice of down-sets of the Morse pre-order, and yields the factorization:

$$\begin{tikzcd}
\mathsf{O}(\mathfrak{X}^{\top}, \leqslant^{\top}) \arrow[r, leftrightarrow, "\cong"] \arrow[dr, leftrightarrow, "\cong"'] & \mathsf{O}(\mathfrak{X}, \leqslant^{\dagger}) \arrow[r, rightarrowtail] \arrow[d, leftrightarrow, "\cong"] & \mathsf{O}(\mathfrak{X}, \leqslant) \arrow[r, rightarrowtail] & \mathsf{Set}(\mathfrak{X}) \\
 & \mathsf{O}(\mathsf{SC}, \leqslant) \arrow[ur, rightarrowtail] \arrow[urr, rightarrowtail, "\mathrm{dyn}^{-1}"'] & &
\end{tikzcd}$$

cf. Thm. 3.28. The discretization $\mathrm{dyn}\colon (\mathfrak{X}, \leqslant) \to (\mathsf{SC}, \leqslant)$ is the map that allows us to alternate between Morse pre-orders and condensed Morse pre-orders , cf. Sect. 3.4, and is dual to the injection $\mathrm{cl}\colon \mathsf{O}(\mathfrak{X}^{\top}, \leqslant^{\top}) \rightarrowtail \mathsf{O}(\mathfrak{X}, \leqslant)$. A formula for dyn is given in Theorem 3.29. In summary, $(\mathfrak{X}, \leqslant^{\dagger})$ is the relevant topology that contains the information about closed attracting blocks. A discrete resolution $(\mathfrak{X}^{\top}, \leqslant^{\top})$ is a coarser data structure than $(\mathfrak{X}, \leqslant^{\dagger})$ and which yields the same down-sets, cf. Fig. 1.3. The Morse pre-order can be retrieved from the condensed Morse pre-order , cf. Thm. 3.19. In Sections 5.1 and 5.3 we exploit this principle in the application to parabolic systems.

[6]For example, CW-decompositions.

[7]Top cells are elements in $(\mathfrak{X}, \leqslant)$ that are maximal with respect to $\leqslant$.

[8]In the application to parabolic systems in Section 5.4 one may achieve a data reduction of orders of magnitude using discrete resolutions, e.g. for a cubical CW-decomposition of a d-cube we have $4^{-d} < |\mathfrak{X}^{\top}|/|\mathfrak{X}| < 2^{-d}$.

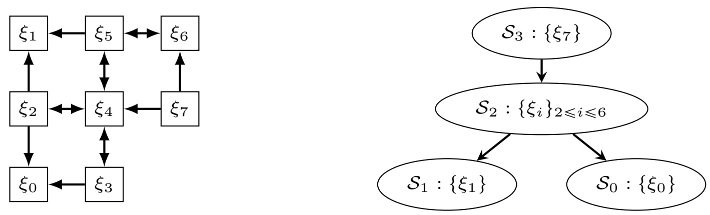

FIGURE 1.3. An associated discrete resolution $\leqslant^{\top}$ on the top-cells $\mathfrak{X}^{\top}$ of CW-decomposition in Fig. 1.1 [left] and the poset SC of its partial equivalence classes [right], which is order-isomorphic to the poset $(\mathsf{T}, \leqslant)$.

1.2. Algebraization

The above constructions have reduced dynamics to a (continuous) discretization $\mathrm{tile}\colon X \to \mathsf{SC}$ of the space X. A discretization as such captures robust directionality properties of a flow. However, information about invariant sets is lost. The third theme in this paper is the algebraization of semi-flows; that is, the order-theoretic structures that encode the directionality are to be augmented with (co)homological data which carry information about (robust) invariant dynamics via Wazewski's principle. A discretization $\mathrm{tile}\colon X \to \mathsf{SC}$, or equivalently a Morse tessellation, gives rise to a filtering of X consisting of regular closed attracting blocks, i.e. a lattice homomorphism $\alpha \mapsto F_\alpha X$, where $\alpha \in \mathsf{O}(\mathsf{SC})$ is a down-set in SC and $F_\alpha X = \mathrm{tile}^{-1}\alpha \in \mathsf{ABlock}_{\mathscr{R}}(\varphi)$ is an attracting block. In the case of homology with field coefficients the representation theory of Cartan-Eilenberg systems, cf. Sect. 4.1, in particular Franzosa's connection matrix theory [19, 20, 21, 31, 58], describes a strict SC-graded chain complex $\big(C^{\mathrm{tile}}(X), \mathrm{d}^{\mathrm{tile}}\big)$ whose grading is given by $C^{\mathrm{tile}}(X) = \bigoplus_{\mathcal{S}\in\mathsf{SC}} H(F_{\downarrow\mathcal{S}}X, F_{\downarrow\mathcal{S}^{\blacktriangleleft}}X)$, where H is the singular homology functor, cf. Fig. 1.4[left], cf. App. C.3. From the graded chain complex $\big(C^{\mathrm{tile}}(X), \mathrm{d}^{\mathrm{tile}}\big)$ all homologies $H(F_\beta X, F_\alpha X)$, with $\alpha, \beta \in \mathsf{O}(\mathsf{SC})$, can be computed as homology of the sub-quotient chain complex $G_{\beta\smallsetminus\alpha}C^{\mathrm{tile}}(X)$ and which is denoted by $H^{\mathrm{tile}}(G_{\beta\smallsetminus\alpha}X)$. This data can be visualized as a poset isomorphic to SC, whose elements are pairs $\big(\mathcal{S}, P^{\mathrm{tile}}_\mu(G_{\mathcal{S}}X)\big) \in \mathsf{SC} \times \mathbb{Z}_+[\mu]$, where $P^{\mathrm{tile}}_\mu(G_{\mathcal{S}}X)$ is the Poincaré polynomial of $H^{\mathrm{tile}}(G_{\mathcal{S}}X)$ which uses the natural dimension grading of singular homology, cf. Fig. 1.4[right]. Such a poset will be referred to as the *tessellar phase diagram*[9] $(\text{Ш}, \leqslant^{\dagger})$ for $(\mathfrak{X}, \leqslant^{\dagger})$, cf. Sect. 4.4.

A discretization $\mathrm{tile}\colon X \to \mathsf{SC}$ may be considered purely from a topological perspective, independent of dynamics. Given a sufficiently 'nice' discretization, the connection matrix theory can be applied to tile. We regard this as part of the synthesis of dynamics and topology, and using dynamical tools to analyze topology. In Section 4.2, we show how this leads to a homology theory which we call *tessellar homology* and which, in contradistinction to cellular homology, uses

[9]The tessellar phase diagram is expressed with respect to Borel-Moore homology.

$$0 \longrightarrow \mathbb{Z}_2\langle \mathcal{S}_2 \rangle \xrightarrow{\mathrm{d}_1^{\mathrm{tile}} = \binom{1}{1}} \bigoplus_{i=\{0,1\}} \mathbb{Z}_2\langle \mathcal{S}_i \rangle \longrightarrow 0$$

$$\begin{tikzcd} & \mathcal{S}_3 : 0 \arrow[d] & \\ & \mathcal{S}_2 : \mu^1 \arrow[dl] \arrow[dr] & \\ \mathcal{S}_1 : \mu^0 & & \mathcal{S}_0 : \mu^0 \end{tikzcd}$$

FIGURE 1.4. The SC-graded chain complex $(C^{\mathrm{tile}}(X), \mathrm{d}^{\mathrm{tile}})$ [left] of Fig. 1.1 (computed using $\mathbb{Z}_2$ coefficients) and associated the tessellar phase diagram $(\text{II}, \leqslant^{\dagger})$ [right]. We display the pair $\big(\mathcal{S}, P_\mu^{\mathrm{tile}}(G_{\mathcal{S}}X)\big)$ as $\mathcal{S}$ and $P_\mu^{\mathrm{tile}}(G_{\mathcal{S}}X)$ as a matter of style.

general tiles instead of CW-cells. In Section 4.3 we show how cellular homology specifically is recovered.

One of the key advantages of using Morse pre-orders is that tile factors through the discretization of $(X, \mathscr{T})$, which enables a computational approach to connection matrix theory using the algorithms of [31, 32] and associated software [33]. For the sake of the simplicity, we explain this in the case that $(X, \mathscr{T})$ admits a finite CW-decomposition. The idea is encapsulated in the following diagram, cf. Sect. 4.3:

$$\begin{tikzcd} & (\mathfrak{X}, \leqslant) \arrow[d, "\mathrm{dyn}"] & \\ X \arrow[ur, "\mathrm{cell}"] \arrow[r, "\mathrm{tile}"'] & \mathsf{SC} \arrow[r, "\mathrm{ind}"'] & \mathbb{Z} \end{tikzcd} \tag{1.2}$$

In the case of a CW-decomposition map cell, the associated pre-order is a partial order (face partial order) and the above diagram provides the factorization via $(\mathfrak{X}, \leqslant)$. The fact that we factor tile through $(\mathfrak{X}, \leqslant)$ allows us to compute $(C^{\mathrm{tile}}(X), \mathrm{d}^{\mathrm{tile}})$ by instead computing connection matrices in two simpler settings. First for the discretization $\mathrm{cell}\colon X \to (\mathfrak{X}, \leqslant)$ to obtain the cellular chain complex $C^{\mathrm{cell}}(X)$ graded over the poset $\mathfrak{X}$ of cells which represents the singular homology as cellular homology; then a second time by re-grading $C^{\mathrm{cell}}(X)$ by SC via $\mathrm{dyn}\colon \mathfrak{X} \to \mathsf{SC}$ and using the algorithm of [31] to produce $(C^{\mathrm{tile}}(X), \mathrm{d}^{\mathrm{tile}})$. As indicated above the discretization $\mathrm{tile}\colon X \to \mathsf{SC}$ is equivalent to a Morse tessellation $(\mathsf{T}, \leqslant) \xrightarrow{\cong} (\mathsf{SC}, \leqslant)$, where the tiles $T \in \mathsf{T}$ are given by $T = \mathrm{tile}^{-1}\mathcal{S}$. Since $\{\mathcal{S}\} = \beta \smallsetminus \alpha$, the homology of T is given by

$$H^{\mathrm{dyn}}(\beta \smallsetminus \alpha) \cong H^{\mathrm{tile}}(G_{\beta \smallsetminus \alpha} X) \cong H^{\mathrm{cell}}(G_{\beta \smallsetminus \alpha} X) \cong H^{\mathrm{BM}}(T),$$

cf. Thm. 4.27, where the latter is the *Borel-Moore homology* of T, cf. [8, 9, 23, 28, 26, 37]. Since T is the set-difference of two attracting blocks for φ it is an isolating block (neighborhood) for φ, cf. [44], and $H^{\mathrm{tile}}(\beta \smallsetminus \alpha)$ represents the Conley index of T.

An additional scalar discretization $\mathrm{ind}\colon \mathsf{SC} \to \mathbb{Z}$, cf. (1.2), allows a second grading of H^{tile} which make H^{tile} a bi-graded homology theory (in the case of field coefficients) denoted by $H^{\mathrm{tile}}_{p,q}(X)$, cf. Sect. 4.2.3. The discretization ind induces a spectral sequence which allows an additional $\mathbb{Z}$-grading of the tessellar homology in accordance with the SC-grading of C^{tile}. As a matter of fact $H^{\mathrm{tile}}_{p,q}(G_{\mathcal{U}\smallsetminus\mathcal{U}'}X)$ is well-defined for any convex set $\mathcal{U} \smallsetminus \mathcal{U}'$ in SC. In this context $(C^{\mathrm{tile}}, \mathrm{d}^{\mathrm{tile}})$ may be regarded as chain complex as well as $\mathbb{Z}$-graded differential module/vector space. Define the Poincaré polynomials $P_{\lambda,\mu}(C^{\mathrm{tile}}) = \sum_{\mathcal{S}\in\mathsf{SC}} P^{\mathrm{tile}}_{\lambda,\mu}(G_{\mathcal{S}}X)$ with $P^{\mathrm{tile}}_{\lambda,\mu}(G_{\mathcal{S}}X) = \sum_{p,q\in\mathbb{Z}}\big(\mathrm{rank}\, H^{\mathrm{tile}}_{p,q}(G_{\mathcal{S}}X)\big)\lambda^p\mu^q$. Then, the following variation on the standard Morse relations are satisfied, cf. Thm. 4.18,

$$\sum_{\mathcal{S}\in\mathsf{SC}} P^{\mathrm{tile}}_{\lambda,\mu}(G_{\mathcal{S}}X) = P^{\mathrm{tile}}_{\lambda,\mu}(X) + \sum_{r=1}^{\infty}(1+\lambda^r\mu)Q^r_{\lambda,\mu},$$

where $Q^r_{\lambda,\mu} \geqslant 0$ and the sum over r is finite. The p-index is a manifestation of the block-flow topology and the q-index of the phase space topology making it a true tale of two topologies. The impact is most apparent in the application to parabolic flows where we use a canonical discretization $\mathrm{lap}\colon \mathsf{SC} \to \mathbb{Z}$.

1.3. Parabolic flows and braid invariants

Chapter 5 encompasses the final theme wherein we demonstrate the ideas and methods in this paper for a large class of flows, called *discrete parabolic flows*. Such flows occur in a wide variety of settings, e.g. studying the infinite dimensional dynamics of scalar parabolic equations which can be realized via discrete parabolic equations, cf. [25], [65]. Discrete parabolic equations and parabolic flows also play a prominent role in the theory of monotone twist maps whose dynamics can be studied using parabolic flows, cf. [2], [14]. The introduction of Morse theory on braids in [24] was sparked by questions for fourth order Lagrangian dynamics which use parabolic flows to describe periodic solutions. Finally, parabolic flows play a pivotal role in computing braid Floer homology, cf. [66].

A parabolic flow φ is defined via differential equations $\dot{x}_i = R_i(x_{i-1}, x_i, x_{i+1})$, with $R_{i+d} = R_i$, where the (smooth) functions $R_i(x_{i-1}, x_i, x_{i+1})$ are monotone with respect to their first and third argument and their stationary equations, given by $R_i(x_{i-1}, x_i, x_{i+1}) = 0$, are referred to as *parabolic recurrence relations*. Parabolic recurrence relations form a perfect symbiosis with discretized braids. A *discretized braid (diagram)* x on n strands and d discretization points is a unordered collection of sequences $\cdots, x^\alpha_0, x^\alpha_1, \cdots$ such that $x^{\theta(\alpha)}_{i+d} = x^\alpha_i$, for a permutation $\theta \in S_n$, and $\alpha = 1, \cdots, n$.

By viewing such sequences as piecewise linear interpolations between the anchor points the various strands 'intersect', cf. Fig. 1.5[left]. There is a non-degeneracy condition on the intersections (no tangencies). A collection of stationary solutions of φ of integer period forms a discretized braid diagrams denoted by

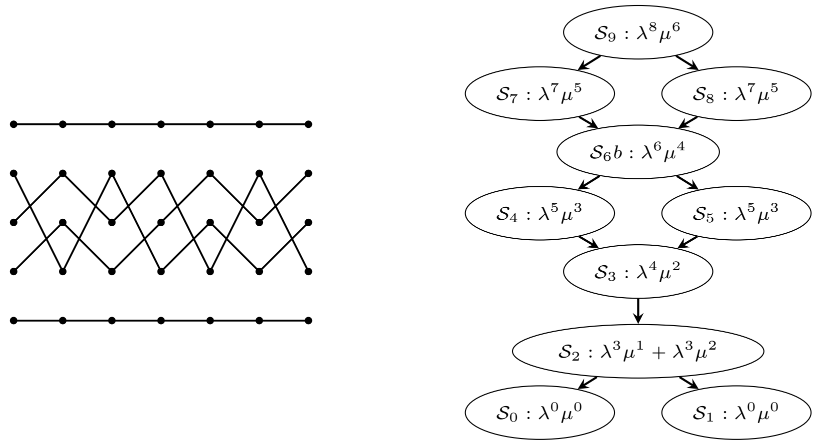

FIGURE 1.5. A discretized (pseudo-anosov) braid y [left], and associated reduced tessellar phase diagram $\overline{\amalg}(y)$ [right]. The vertices in $\overline{\amalg}(y)$ contain the Poincaré polynomials of the bi-graded parabolic homology, cf. Sect. 5.4.2.

y and referred to as a *skeleton*, cf. Fig. 1.5[left]. If we add an additional periodic sequence x the ensemble of x and y generically forms a braid $x \operatorname{rel} y$, called a *relative* discretized braid. Since y is stationary for φ we may consider $\varphi(t,x) \operatorname{rel} y$. Whenever $\varphi(t,x) \operatorname{rel} y$ becomes singular, i.e. strands develop tangencies, then the total number of intersections of x with the strands in y decreases strictly. This principle is crucial and emphasizes the intimate relation between parabolic flows and discretized braids. As such relative braids can be partially ordered using parabolic flows which form the backbone of the canonical discretization of the block-flow topology for parabolic flows. We construct special CW-decompositions, cf. Sect. 5, and discretizations of the block-flow topology via appropriately defined discrete Lyapunov functions, which allows us to establish Morse tessellations and Morse representations, cf. Sect. 5.3. In the language of condensed Morse pre-orders the partial order on the relative braid classes defines the poset $(\mathsf{SC}, \leqslant)$ which comes from the canonical CW-decomposition of X given by the skeleton y and the block-flow topology given by φ.

In Section 5.4.1 we give an overview of the algorithmic steps in computing the order structures and connection matrices and in Section 5.4.2 we discuss the bi-graded parabolic homology. In Section 5.5 we show that the parabolic differential module (and the associated tessellar phase diagrams) we obtain for parabolic flows extend the results in [24] and provide an invariant for positive braids which also defines a new invariant for scalar parabolic equations, cf. Thm. 5.23. The importance of Theorem 5.23 is that we obtain complete insight in the homology of loop spaces for scalar parabolic equations as well as the boundary homomorphisms which contain information about connecting orbits for parabolic equations. To best explain the discretization of topology and dynamics and the main

$$0 \longrightarrow \mathbb{Z}_2\langle\mathcal{S}_9\rangle \xrightarrow{\mathrm{d}_6^{\mathrm{tile}} = \begin{pmatrix}1\\1\end{pmatrix}} \bigoplus_{i=7,8} \mathbb{Z}_2\langle\mathcal{S}_i\rangle \xrightarrow{\mathrm{d}_5^{\mathrm{tile}} = \begin{pmatrix}1 & 1\end{pmatrix}} \mathbb{Z}_2\langle\mathcal{S}_6\rangle \xrightarrow{\mathrm{d}_4^{\mathrm{tile}} = \begin{pmatrix}0\\0\end{pmatrix}}$$

$$\bigoplus_{i=4,5} \mathbb{Z}_2\langle\mathcal{S}_i\rangle \xrightarrow{\mathrm{d}_3^{\mathrm{tile}} = \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}} \bigoplus_{i=2,3} \mathbb{Z}_2\langle\mathcal{S}_i\rangle \xrightarrow{\mathrm{d}_2^{\mathrm{tile}} = \begin{pmatrix}0 & 0\end{pmatrix}} \mathbb{Z}_2\langle\mathcal{S}_2\rangle \xrightarrow{\mathrm{d}_1^{\mathrm{tile}} = \begin{pmatrix}1\\1\end{pmatrix}} \bigoplus_{i=0,1} \mathbb{Z}_2\langle\mathcal{S}_i\rangle \xrightarrow{\mathrm{d}_0^{\mathrm{tile}} = 0} 0$$

FIGURE 1.6. Associated complex $\mathscr{A}(\beta)$, computed over $\mathbb{K} = \mathbb{Z}_2$ coefficients, viewed as chain complex using dimension grading. The index q for $\mathrm{d}_q^{\mathrm{tile}}$ reflects the natural grading induced by singular homology.

statement of Theorem 5.23 we consider an example. Let y be a discretized braid y given in Figure 1.5[left] and φ is any parabolic flow for which y is stationary. This can be rephrased in terms of a bi-topological space for which y defines a natural discretization. The sub-poset of homologically non-trivial braid classes in $(\mathsf{SC}, \leqslant)$ is given by the reduced tessellar phase diagram $(\overline{\amalg}, \leqslant^{\dagger})$ displayed in Fig. 1.5[right]. The more detailed information is given by the *parabolic differential module* $\mathscr{A}(\beta)$ generated by $\mathrm{tile}\colon X \to \mathsf{SC}$ and the scalar discretization $\mathrm{lap}\colon \mathsf{SC} \to \mathbb{Z}$, which counts the intersections of x with y divided by two, cf. Sect. 5.5 for a detailed definition. As described in Section 5.4.2 lap provides a scalar grading on the tessellar homology $H^{\mathrm{tile}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'}X)$, with $\mathcal{U}\smallsetminus\mathcal{U}'$ convex in SC, which yields the *parabolic homology* $\vec{H}_{p,q}(G_{\mathcal{U}\smallsetminus\mathcal{U}'}X)$ where

$$H^{\mathrm{tile}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'}X) = \bigoplus_{p,q\in\mathbb{Z}} \vec{H}_{p,q}(G_{\mathcal{U}\smallsetminus\mathcal{U}'}X).$$

Figure 1.5[right] lists the Poincaré polynomials of the non-trivial tiles with respect to parabolic homology. The differential module $\mathscr{A}(\beta)$ may be regarded as a chain complex in Figure 1.6 with homology $H_q^{\mathrm{BM}}(G_{\mathcal{S}}X) = \oplus_{p\in\mathbb{Z}}\vec{H}_{p,q}(G_{\mathcal{S}}X)$, or as $\mathbb{Z}$-graded differential module in Figure 1.7 with homology $\vec{H}_{p,*}(G_{\mathcal{S}}X) = \oplus_{q\in\mathbb{Z}}\vec{H}_{p,q}(G_{\mathcal{S}}X)$. By making further specification of the entries in $\mathrm{d}^{\mathrm{tile}}$ the differential module can also be represented as to show the $\overline{\amalg}$-order, cf. Sect. 5.5. Theorem 5.23 shows that the parabolic differential module, and thus the reduced tessellar phase diagram is a topological invariant for the topological braid $\beta(y)$, i.e. all braid diagrams isotopic to y, cf. Sect. 5.5. Due to Theorem 5.23 the parabolic differential module and reduced tessellar phase diagram can be denoted by $\mathscr{A}(\beta)$ and $\overline{\amalg}(\beta)$ respectively. In summary, the application of the methods put forth in this paper to parabolic flows gives a novel approach towards computing algebraic topological information about infinite dimensional problems.

The themes throughout this paper touch upon many topics. Section 6 concludes with a discussion of related remarks and open problems.

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2\langle\mathcal{S}_9\rangle \arrow[r, "\mathrm{d}_8^{\mathrm{tile}} = \begin{pmatrix}1\\1\end{pmatrix}"] & \bigoplus\limits_{i=7,8} \mathbb{Z}_2\langle\mathcal{S}_i\rangle \arrow[r, "\mathrm{d}_7^{\mathrm{tile}} = \begin{pmatrix}1 & 1\end{pmatrix}"] & \mathbb{Z}_2\langle\mathcal{S}_6\rangle \arrow[dlll, rounded corners, "\mathrm{d}_6^{\mathrm{tile}} = \begin{pmatrix}0\\0\end{pmatrix}"] & \\
\bigoplus\limits_{i=4,5} \mathbb{Z}_2\langle\mathcal{S}_i\rangle \arrow[r, "\mathrm{d}_5^{\mathrm{tile}} = \begin{pmatrix}1 & 1\end{pmatrix}"] \arrow[rr, bend right, "\mathrm{d}_5^{\mathrm{tile}} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}"'] & \mathbb{Z}_2\langle\mathcal{S}_3\rangle \arrow[r, "\mathrm{d}_4^{\mathrm{tile}} = \begin{pmatrix}0\\0\end{pmatrix}"] & \mathbb{Z}_2 \oplus \mathbb{Z}_2\langle\mathcal{S}_2\rangle \arrow[r, "\mathrm{d}_3^{\mathrm{tile}} = \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}"] & \bigoplus\limits_{i=0,1} \mathbb{Z}_2\langle\mathcal{S}_i\rangle \arrow[r] & 0
\end{tikzcd}
$$

FIGURE 1.7. The differential module $\mathscr{A}(\beta)$, computed over $\mathbb{K} = \mathbb{Z}_2$ coefficients, with lap number grading. For the entry $\mathrm{d}_p^{\mathrm{tile}}$ the index p reflects the lap number.

CHAPTER 2

Topology, discretization and bi-topological spaces

Our philosophy is that discretization is the study of appropriate *finite* substructures. In this chapter we start with an exposition of topological spaces, closure algebras and modal algebras and discuss discretization in terms of Boolean algebras with operators. Closure algebras provide an equivalent way to describe a topological space. In general modal algebras are also related to topological spaces which provides the essential link to dynamical systems. The latter can be used to regard various aspects of dynamics in terms of topology.

2.1. Closure algebras

Let $(X, \mathscr{T})$ be a topological space and let $\mathsf{Set}(X)$ denote the (complete and atomic) Boolean algebra of subsets of X. For $(X, \mathscr{T})$ we define an associated *closure algebra* as the pair $\big(\mathsf{Set}(X), \mathrm{cl}\big)$, where $\mathrm{cl}\colon \mathsf{Set}(X) \to \mathsf{Set}(X)$ is the operator defined as the closure of a subset, $\mathrm{cl}\, U = \bigcap\big\{U' \supset U \mid U' \text{ closed}\big\}$, cf. [51], which is our first source of closure algebras. In general, an operator $\mathrm{cl}\colon \mathsf{Set}(X) \to \mathsf{Set}(X)$ is a *closure operator* if all four Kuratowski axioms for a closure operator are satisfied: for all $U, U' \subset X$,

(K1) (normal) $\mathrm{cl}\, \varnothing = \varnothing$;
(K2) (additive) $\mathrm{cl}(U \cup U') = \mathrm{cl}\, U \cup \mathrm{cl}\, U'$;
(K3) (sub-idempotent) $\mathrm{cl}\big(\mathrm{cl}\, U\big) \subset \mathrm{cl}\, U$;[1]
(K4) (expansive) $U \subset \mathrm{cl}\, U$.[2]

Continuous maps between topological spaces can also be described in terms of closure algebras. Let $g\colon X \to Y$ be a continuous map between topological spaces. Then, g^{-1} defines a map from $\mathsf{Set}(Y)$ to $\mathsf{Set}(X)$. As a matter of fact $g^{-1}\colon \mathsf{Set}(Y) \to \mathsf{Set}(X)$ is a completely additive[3] Boolean homomorphism of complete and atomic

[1]Axiom (K3) in combination with Axiom (K4) this implies that cl is idempotent, i.e. $\mathrm{cl}\big(\mathrm{cl}\, U\big) = \mathrm{cl}\, U$.

[2]The single condition (K): $U \cup \mathrm{cl}\, U \cup \mathrm{cl}\big(\mathrm{cl}\, U'\big) = \mathrm{cl}(U \cup U') \smallsetminus \mathrm{cl}\, \varnothing$ is equivalent to (K1)-(K4).

[3]Closed with respect to arbitrary intersections and unions.

Boolean algebras, cf. App. A.3. Consider the (not necessarily commutative) diagram:

$$
\begin{tikzcd}
\mathsf{Set}(X) \arrow[r, "\mathrm{cl}_X"] & \mathsf{Set}(X) \\
\mathsf{Set}(Y) \arrow[u, "g^{-1}"] \arrow[r, "\mathrm{cl}_Y"] & \mathsf{Set}(Y) \arrow[u, "g^{-1}"']
\end{tikzcd}
\tag{2.1}
$$

The continuity of g is equivalent to the condition $\mathrm{cl}_X g^{-1}(V) \subset g^{-1}(\mathrm{cl}_Y V)$ for all $V \subset Y$ which makes g^{-1} a *semi-homomorphism* of closure algebras. In particular, if $V \subset Y$ is closed in Y, then $g^{-1}(V)$ is closed in X. In case $\mathrm{cl}_X g^{-1}(V) = g^{-1}(\mathrm{cl}_Y V)$ for all $V \subset Y$ the operator g^{-1} is called a homomorphism of closure algebras,[4] in which case g is an open continuous map.[5] A second source for closure algebras are via pre-ordered sets. Let $(X, \leqslant)$ be a pre-order. Define

$$\mathrm{cl}_{\leqslant} U := \downarrow U = \{x \in X \mid x \leqslant y \text{ for some } y \in U\}.$$

Then, $(\mathsf{Set}(X), \mathrm{cl}_{\leqslant})$ is a closure algebra and the closure operator $\mathrm{cl}_{\leqslant}$ is completely additive, i.e. Kuratowski axiom (K2) is satisfied with respect to arbitrary unions. The associated topological space is denoted by $(X, \mathscr{T}_{\leqslant})$ which is an Alexandrov topological space and the topology $\mathscr{T}_{\leqslant}$ is called an *Alexandrov topology*, i.e. $\mathscr{T}_{\leqslant}$ is closed under arbitrary intersections and unions.

On the other hand every topological space induces a natural pre-order as follows:

$$x \leqslant_{\mathscr{T}} x' \quad \text{if and only if} \quad x \in \mathrm{cl}\{x'\},$$

which is called the *specialization pre-order* associated to $(X, \mathscr{T})$. In general, the topology $\mathscr{T}_{\leqslant_{\mathscr{T}}}$ induced by $\leqslant_{\mathscr{T}}$ is finer than $\mathscr{T}$. In particular $\downarrow U$ is not equal to $\mathrm{cl}\, U$ in that case. If we start from an Alexandrov topology $\mathscr{T}$, then $\mathscr{T}_{\leqslant_{\mathscr{T}}} = \mathscr{T}$. The above described duality between topological spaces and pre-orders will be used to treat discretization of topological spaces, cf. [5] for further details on closure algebras. A topological space yields a closure algebra where the Boolean algebra is complete and atomic. This concept can be defined for any Boolean algebra, cf. Sect. 2.6.3 and App. A.

2.2. Modal operators and modal algebras

A third source for closure algebras is given by modal operators and binary relations on a set X, cf. App. A. In general, an operator $\Phi\colon \mathsf{Set}(X) \to \mathsf{Set}(X)$ is called a *modal operator* if following axioms are satisfied: for all $U, U' \subset X$,

(M1) (normal) $\Phi\varnothing = \varnothing$;
(M2) (additive) $\Phi(U \cup U') = \Phi U \cup \Phi U'$;

---

[4] For a *homomorphism* of closure algebras the diagram in (2.1) commutes.

[5] If there is no ambiguity about the topological space the sub-index of cl will be omitted.

The pair $(\mathsf{Set}(X), \Phi)$ is called a *modal algebra* and is an example of a Boolean algebra with operators, cf. [38], [51]. A modal algebra defines a topology on X. Consider the set $\mathsf{Fwdset}(\Phi)$ consisting of subsets $U \subset X$ such that $\Phi U \subset U$.

PROPOSITION 2.1. *The set* $\mathsf{Fwdset}(\Phi)$ *defines a bounded, distributive lattice with binary operations* $\cap$ *and* $\cup$. *Moreover,* $\mathsf{Fwdset}(\Phi)$ *is closed under arbitrary intersections, i.e. arbitrary intersections of sets in* $\mathsf{Fwdset}(\Phi)$ *are again in* $\mathsf{Fwdset}(\Phi)$.

PROOF. The subsets $\varnothing$ and X are in $\mathsf{Fwdset}(\Phi)$ since Φ is a normal operator. Finite unions of sets in $\mathsf{Fwdset}(\Phi)$ are obviously again in $\mathsf{Fwdset}(\Phi)$ since Φ is additive. Let $\{U_i\}$ be an arbitrary collection is subsets in $\mathsf{Fwdset}(\Phi)$. Then, $\Phi\big(\bigcap_i U_i\big) \subset \Phi U_i \subset U_i$ and therefore $\Phi\big(\bigcap_i U_i\big) \subset \bigcap_i U_i$. $\square$

The lattice $\mathsf{Fwdset}(\Phi)$ defines a topology $\mathscr{T}_\Phi$ on X by declaring the subsets in $\mathsf{Fwdset}(\Phi)$ to be the closed sets. The topology $\mathscr{T}_\Phi$ given by the lattice $\mathsf{Fwdset}(\Phi)$ can also be characterized by an associated closure operator.

PROPOSITION 2.2. *Consider the operator* $\mathrm{cl}_\Phi \colon \mathsf{Set}(X) \to \mathsf{Set}(X)$ *defined by*

$$\mathrm{cl}_\Phi U = \bigcap \big\{ U' \supset U \mid U' \in \mathsf{Fwdset}(\Phi) \big\} \in \mathsf{Fwdset}(\Phi). \tag{2.2}$$

Then, cl_Φ *is a closure operator and* $\mathsf{Fwdset}(\Phi) = \mathsf{Fwdset}(\mathrm{cl}_\Phi)$, *i.e. the sets in* $\mathsf{Fwdset}(\Phi)$ *are exactly the closed sets defined by* cl_Φ.

PROOF. The definition of cl_Φ is a standard construction of a closure operator satisfying the Kuratowski axioms (K1)-(K4). A set $U \in \mathsf{Fwdset}(\mathrm{cl}_\Phi)$ satisfies $\mathrm{cl}_\Phi U \subset U$ and thus $\mathrm{cl}_\Phi U = U$, i.e. $U \in \mathsf{Fwdset}(\Phi)$ and thus $\mathsf{Fwdset}(\mathrm{cl}_\Phi) \subset \mathsf{Fwdset}(\Phi)$. If $U \in \mathsf{Fwdset}(\Phi)$, then $\mathrm{cl}_\Phi U = U$ which yields $\mathsf{Fwdset}(\Phi) \subset \mathsf{Fwdset}(\mathrm{cl}_\Phi)$. Combining both inclusions gives the desired statement. $\square$

The lattice $\mathsf{Fwdset}(\Phi)$ is a complete *co-Heyting algebra* with binary subtraction $U - U' := \mathrm{cl}_\Phi(U \smallsetminus U')$. The specialization pre-order $\leqslant_\Phi$ defined by the topology $\mathscr{T}_\Phi$ (or equivalently the closure operator cl_Φ) is the transitive reflexive closure of the binary relation $\phi \subset X \times X$, referred to as the *specialization relation*, which is defined by

$$(x, x') \in \phi \quad \text{if and only if} \quad x \in \Phi\{x'\}, \tag{2.3}$$

where $\Phi = \phi^{-1}$ is regarded as an operator on $\mathsf{Set}(X)$ and is an example of a completely additive modal operator, cf. A.3. As before $\downarrow U$ does not coincide with $\mathrm{cl}_\Phi U$ in general. This is due to the fact that a modal operator is not completely additive[6] in general. For the time being the above described duality between $(X, \mathscr{T}_\Phi)$ and the associated closure algebra $(\mathsf{Set}(X), \mathrm{cl}_\Phi)$ suffices.

In the case Φ is a *completely additive* modal operator on $\mathsf{Set}(X)$ the specialization relation recovers Φ and vice versa. To be more precise,

$$(x, x') \in \phi \quad \text{if and only if} \quad (x', x) \in \phi^{-1} \quad \text{if and only if} \quad x \in \phi^{-1}\{x'\},$$

[6] An operator is said to be completely additive if it is closed under arbitrary unions.

where for the latter we regard the opposite relation ϕ^{-1} as modal operator:

$$\Phi U := \bigcup_{x\in U} \Phi\{x\}, \quad \text{where} \quad \Phi\{x\} = \big\{y \in X \mid (x,y) \in \phi^{-1} \text{ for some } x \in U\big\}. \tag{2.4}$$

The transitive reflexive closure $\phi^{+=} = \bigcup_{k\geqslant 0} \phi^k$ defines a pre-order $(X, \phi^{+=})$ and therefore $\Phi^{+=} = \bigcup_{k\geqslant 0} \Phi^k$ is a completely additive closure operator. In particular:

LEMMA 2.3. *$\Phi^{+=}U = \downarrow U = \mathrm{cl}_\Phi U$.*

PROOF. The first equility is a direct consequence of the duality between ϕ and Φ. As for the second equality we argue as follows. If $\mathrm{cl}_\Phi U = U$, then $U \in \mathsf{Fwdset}(\Phi)$, i.e. $\Phi U \subset U$ and thus $\Phi^{+=}U = U$. Conversely, if $\Phi^{+=}U = U$, then $\Phi U \subset U$ and thus $\mathrm{cl}_\Phi U = U$. Therefore, $\Phi^{+=}U = U$ if and only if $\mathrm{cl}_\Phi U = U$ and $\Phi^{+=}U = U = \mathrm{cl}_\Phi U$. □

In this case a binary relation $\phi \subset X \times X$ is the source of a closure algebra: $\big(\mathsf{Set}(X), \Phi^{+=}\big)$. The associated topology is an Alexandrov topology and is equivalent to the specialization pre-order, cf. A.1.

Let $(\mathsf{Set}(X), \Phi)$ and $(\mathsf{Set}(Y), \Psi)$ be modal algebras. As for closure algebras a map $g\colon X \to Y$ yields a Boolean homomorphism $g^{-1}\colon \mathsf{Set}(Y) \to \mathsf{Set}(X)$. The latter is a *semi-homomorphism of modal algebras* if $\Phi g^{-1}(V) \subset g^{-1}(\Psi V)$ for all $V \subset Y$ and is expressed in the (not necessarily commutative) diagram:

$$\begin{tikzcd} \mathsf{Set}(X) \arrow[r, "\Phi"] & \mathsf{Set}(X) \\ \mathsf{Set}(Y) \arrow[u, "g^{-1}"] \arrow[r, "\Psi"] & \mathsf{Set}(Y) \arrow[u, "g^{-1}"'] \end{tikzcd} \tag{2.5}$$

The semi-homomorphism property for Boolean homomorphisms $g^{-1}\colon \mathsf{Set}(Y) \to \mathsf{Set}(X)$ is related to continuity of g:

PROPOSITION 2.4. *Let $g^{-1}\colon (\mathsf{Set}(Y), \Phi) \to (\mathsf{Set}(X), \Psi)$ be a semi-homomorphism of modal algebras for a map $g\colon X \to Y$, i.e. $\Phi g^{-1}(V) \subset g^{-1}(\Psi V)$ for all $V \subset Y$. Then, $g\colon (X, \mathscr{T}_\Phi) \to (Y, \mathscr{T}_\Psi)$ is a continuous map.*

PROOF. The closure operators cl_Ψ and cl_Ψ are given by (2.2). Since g^{-1} is a completely additive Boolean homomorphism we have

$$g^{-1}\big(\mathrm{cl}_\Psi V\big) = \bigcap \big\{g^{-1}(V') \mid V' \supset V,\ \Psi V' \subset V'\big\}.$$

The fact that g^{-1} is a semi-homomorphism of modal algebras implies, for V' closed in Y, that $\Phi g^{-1}(V') \subset g^{-1}(\Psi V') \subset g^{-1}(V')$ and thus $g^{-1}(V')$ is closed in X. This implies that $\mathrm{cl}_\Phi g^{-1}(V) \subset g^{-1}(\mathrm{cl}_\Psi V)$, which proves that $g\colon X \to Y$ is a continuous map. □

REMARK 2.5. If $\Xi\colon \mathsf{Set}(Y) \to \mathsf{Set}(X)$ is completely additive Boolean homomorphism then $\Xi = g^{-1}$ for some map $g\colon X \to Y$, cf. Prop. A.7 and [38]. The latter is given by $g(x) = y$ for the unique $y \in Y$ such that $x \in \Xi\{y\}$.

REMARK 2.6. If Φ and Ψ are completely additive operators then Proposition 2.4 can be proved using transitive reflexive closure. Observe that $\Phi^k g^{-1}(V) \subset g^{-1}(\Psi^k V)$ for all $k \geqslant 0$. Then,

$$\mathrm{cl}_\Phi g^{-1}(V) = \bigcup_{k\geqslant 0} \Phi^k g^{-1}(V) \subset \bigcup_{k\geqslant 0} g^{-1}(\Psi^k V) = g^{-1}\big(\bigcup_{k\geqslant 0} \Psi^k V\big) = g^{-1}(\mathrm{cl}_\Psi V),$$

which establishes continuity.

REMARK 2.7. For a pre-order $(X, \leqslant)$ the associated dual given by $\mathrm{cl}_\leqslant U = \downarrow U$ defines a *complete and atomic and completely additive* closure algebra $(\mathsf{Set}(X), \mathrm{cl}_\leqslant)$, i.e. $\mathsf{Set}(X)$ is a complete and atomic Boolean algebra and the closure operator $\mathrm{cl}_\leqslant$ is completely additive. The completely additive closure operator $\mathrm{cl}_\leqslant$ retrieves the pre-order. Similarly, a binary relation $\phi \subset X \times X$ yields a complete and atomic, and completely additive closure algebra $(\mathsf{Set}(X), \mathrm{cl}_\Phi)$, where $\mathrm{cl}_\Phi = \Phi^{+=}$ and $\Phi = \phi^{-1}$. The closure operator retrieves the transitive reflexive relation $\phi^{+=}$, but not ϕ in general. For a complete and atomic, and completely additive modal algebra $(\mathsf{Set}(X), \Phi)$ the operator Φ retrieves ϕ. These principles play a role in the duality theory of closure algebras and modal algebras, cf. App. A.3, Sect. 2.6.3 and [51], [38]. In this text we are mainly interested in closure algebras and their duality.

REMARK 2.8. Closely related to a closure operator is the notion of a derivative operator. A modal operator $\Gamma\colon \mathsf{Set}(X) \to \mathsf{Set}(X)$ is called a *derivative operator* if following axioms are satisfied: for all $U, U' \subset X$,

(D1) (normal) $\Gamma\varnothing = \varnothing$;
(D2) (additive) $\Gamma(U \cup U') = \Gamma U \cup \Gamma U'$;
(D3) (quasi-idempotent) $\Gamma(\Gamma U) \subset U \cup \Gamma U$.[7]

The pair $(\mathsf{Set}(X), \Gamma)$ is called a *derivative algebra*, cf. [17], [51]. A derivative operator defines a closure operator via

$$\mathrm{cl}_\Gamma U := U \cup \Gamma U. \tag{2.6}$$

For every closure operator there exists a derivative operator such that (2.6) holds, e.g. take $\Gamma = \mathrm{cl}$.[8]

2.3. Discretization of topology

We start with a general description of discretization of topology in terms of closure algebras. This procedure can then be used for the same purpose in the setting of modal algebras. These techniques play a role in the discussion of treating dynamics in terms of topology.

[7] Axiom (D3) is equivalent to from Axiom (K4) via $\mathrm{cl} = \mathrm{id} \cup \Gamma$.

[8] The choice of a derivative operator is clearly not unique. An important non-trivial choice is given by the *derived set*, the set of limit points of a set U:

$$\Gamma U := \{x \in X \mid N \cap U \smallsetminus \{x\} \neq \varnothing \text{ for all neighborhoods } N \ni x\},$$

which is *not* equal to $\mathrm{cl}\, U$ in general.

2.3.1. Closure algebra discretization. Discretization of a topological space $(X,\mathscr{T})$ in the spirit of closure algebras is an *injective* Boolean homomorphism[9] $|\cdot|\colon \mathsf{Set}(\mathfrak{X}) \rightarrowtail \mathsf{Set}(X)$, where $\mathsf{Set}(\mathfrak{X})$ is the powerset of a finite set $\mathfrak{X}$, in combination with an appropriately chosen *discrete* closure operator $\mathbf{cl}\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$ such that $\mathrm{cl}|\mathcal{U}| \subset |\mathbf{cl}\mathcal{U}|$ for all $\mathcal{U} \in \mathsf{Set}(\mathfrak{X})$, in which case $|\cdot|\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(X)$ is a semi-homomorphism of closure algebras. We refer to the elements $\xi \in \mathfrak{X}$ as *cells*. This discretization is captured by the following diagram in the category of closure algebras and semi-homomorphisms:[10]

$$
\begin{tikzcd}
\mathsf{Set}(X) \arrow[r, "\mathrm{cl}"] & \mathsf{Set}(X) \\
\mathsf{Set}(\mathfrak{X}) \arrow[u, tail, "|\cdot|"] \arrow[r, "\mathbf{cl}"] & \mathsf{Set}(\mathfrak{X}) \arrow[u, tail, "|\cdot|"']
\end{tikzcd}
\tag{2.7}
$$

The closure algebra $(\mathsf{Set}(\mathfrak{X}),\mathbf{cl})$ defines a finite topology on $\mathfrak{X}$ by declaring $\mathcal{U}$ closed if and only if $\mathbf{cl}\mathcal{U} = \mathcal{U}$. As $\mathfrak{X}$ is a finite set any topology on $\mathfrak{X}$ is necessarily an Alexandrov topology, and is equivalent to the specialization pre-order $(\mathfrak{X},\leqslant)$[11] defined by

$$
\xi \leqslant \xi' \quad \text{if and only if} \quad \xi \in \mathrm{cl}\{\xi'\},
\tag{2.8}
$$

cf. App. B.3. The discretization described above allows us to regard $\mathfrak{X}$ as an algebra $\big(\mathsf{Set}(\mathfrak{X}),\mathbf{cl}\big)$, as a pre-order $(\mathfrak{X},\leqslant)$, and as topological space $\mathfrak{X}$. In general we do not differentiate between the specialization pre-order and the Alexandrov topology, and we refer to the triple $(\mathfrak{X},\mathbf{cl},|\cdot|)$ as a closure algebra discretization, or *CA-discretization* of $(X,\mathscr{T})$.

DEFINITION 2.9. A CA-discretization is called *Boolean* if $\mathrm{cl}|\mathcal{U}| = |\mathbf{cl}\mathcal{U}|$ for all $U \in \mathsf{Set}(\mathfrak{X})$.

This definition in particular implies that (2.7) is commutative in which case the map $|\cdot|\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(X)$ is a homomorphism of closure algebras, cf. [5]. For a pre-order $(\mathfrak{X},\leqslant)$ we define a *down-set* $\mathcal{U} \subset \mathfrak{X}$ by the property: $\xi' \in \mathcal{U}$, $\xi \leqslant \xi'$, then $\xi \in \mathcal{U}$. The set of down-sets is denoted by $\mathsf{O}(\mathfrak{X},\leqslant)$[12] which by construction is a finite distributive lattice with binary operations $\cap$ and $\cup$, cf. App. B.1, B.3 and [15]. Note that $\mathsf{O}(\mathfrak{X},\leqslant) = \mathsf{Fwdset}(\mathbf{cl})$, where $\mathbf{cl}$ is the associated closure operator on $\mathsf{Set}(\mathfrak{X})$. In a similar fashion we can define the lattice of *up-sets* $\mathsf{U}(\mathfrak{X},\leqslant) = \mathsf{Fwdset}(\bar{\mathbf{c}}\mathbf{l})$, where

[9] In the context of discretizaztion we consider injective homomorphisms with finite range, i.e. finite subalgebras. The theory can be phrased in more general terms via homomorphisms.

[10] In the above mentioned category of closure algebras we employ the morphisms are semi-homomorphisms of closure algebras.

[11] The Alexandrov topology is T_0 if and only if the specialization pre-order is a partial order.

[12] If there is no ambiguity about the pre-order we write $\mathsf{O}(\mathfrak{X})$ for short. Another common notation is $\mathsf{Invset}^+(\leqslant)$.

$\bar{\mathbf{cl}}\,\mathcal{U} = \mathbf{star}\,\mathcal{U}$ is the *conjugate closure operator*, cf. App. A.2 and [38]. The join-irreducible elements of both lattices are characterized as

$$\mathbf{cl}\,\xi = \downarrow\xi := \{\xi' \mid \xi' \leqslant \xi\} \quad \text{and} \quad \mathbf{star}\,\xi = \uparrow\xi := \{\xi' \mid \xi \leqslant \xi'\}, \quad \xi \in \mathcal{X},$$

which are called the *principal* down-sets and up-sets respectively. Intersections of up-sets and down-sets are the *convex sets* in $(\mathcal{X}, \leqslant)$[13] and are denoted by $\mathsf{Co}(\mathcal{X}, \leqslant)$ which is a meet-semilattice with respect to $\cap$.

REMARK 2.10. For a pre-order $(\mathcal{X}, \leqslant)$ we can define the *partial equivalence classes* by $\xi \sim \xi'$ if and only if $\xi \leqslant \xi'$ and $\xi' \leqslant \xi$. The set of partial equivalence classes is denoted by $\mathcal{X}/_\sim$. The latter is a poset via $[\xi] \leqslant [\eta]$ if only only if $\xi \leqslant \eta$. This yields the natural order-preserving projection $\mathcal{X} \xrightarrow{\pi} \mathcal{X}/_\sim$ defined by $\xi \mapsto [\xi]$. By construction $\mathsf{O}(\mathcal{X}, \leqslant) \cong \mathsf{O}(\mathcal{X}/_\sim)$. The map $X \twoheadrightarrow \mathcal{X} \twoheadrightarrow \mathcal{X}/_\sim$ is also a discretization. The associated CA-discretization $(\mathcal{X}/_\sim, \mathbf{cl}, |\cdot|)$ is defined by $\mathrm{cl}[\xi] = \downarrow[\xi]$ and $\big|[\xi]\big| = \bigcup_{\xi' \in [\xi]} |\xi'|$ and yields a T_0 Alexandrov topology.

2.3.2. Modal algebra discretization. Let $(X, \mathscr{T}_\Phi)$ be a topological space defined by a modal operator $\Phi\colon \mathsf{Set}(X) \to \mathsf{Set}(X)$. The associated closure algebra is $(\mathsf{Set}(X), \mathrm{cl}_\Phi)$ with closure operator defined in (2.2). Discretization in terms of closure algebras can always be formulated in terms of modal algebras. Consider a discrete modal operator $\mathbf{\Phi}\colon \mathsf{Set}(\mathcal{X}) \to \mathsf{Set}(\mathcal{X})$. As explained in Section 2.2 we obtain a topological space $(\mathcal{X}, \mathscr{T}_{\mathbf{\Phi}})$, whose discrete topology can be described by either the specialization pre-order $\leqslant_{\mathbf{\Phi}}$ or the associated closure operator $\mathbf{cl}_{\mathbf{\Phi}}$. A triple $(\mathcal{X}, \mathbf{\Phi}, |\cdot|)$ is a modal algebra discretization, or *MA-discretization* of $(X, \mathscr{T}_\Phi)$ if

(MA) $\Phi|\mathcal{U}| \subset |\mathbf{\Phi}\mathcal{U}|$ for all $\mathcal{U} \in \mathsf{Set}(\mathcal{X})$,

which is equivalent to the condition that $|\cdot|$ is a semi-homomorphism of modal algebras and is expressed in the diagram:

$$\begin{tikzcd} \mathsf{Set}(X) \arrow[r, "\Phi"] & \mathsf{Set}(X) \\ \mathsf{Set}(\mathcal{X}) \arrow[r, "\mathbf{\Phi}"] \arrow[u, "|\cdot|"] & \mathsf{Set}(\mathcal{X}) \arrow[u, "|\cdot|"'] \end{tikzcd} \tag{2.9}$$

PROPOSITION 2.11. *Let $(\mathcal{X}, \mathbf{\Phi}, |\cdot|)$ be a MA-discretization of $(X, \mathscr{T}_\Phi)$. Then, the induced closure operator $\mathbf{cl}_{\mathbf{\Phi}}\colon \mathsf{Set}(\mathcal{X}) \to \mathsf{Set}(\mathcal{X})$ given in (2.2) defines a CA-discretization $(\mathcal{X}, \mathbf{cl}_{\mathbf{\Phi}}, |\cdot|)$ of X. In particular, the closure operator $\mathbf{cl}_{\mathbf{\Phi}}$ is given by $\mathbf{cl}_{\mathbf{\Phi}} = \bigcup_{k\geqslant 0} \mathbf{\Phi}^k$.*[14]

PROOF. Since $|\cdot|$ is a completely additive Boolean homomorphism it is the inverse image of a map $X \to \mathcal{X}$, cf. Prop. A.7 and Rem. 2.5. By Axiom (MA) and Proposition 2.4 the latter is continuous and thus $\mathrm{cl}_\Phi|\mathcal{U}| \subset |\mathbf{cl}_{\mathbf{\Phi}}\mathcal{U}|$ for all $\mathcal{U} \subset \mathcal{X}$, which proves that $(\mathcal{X}, \mathbf{cl}_{\mathbf{\Phi}}, |\cdot|)$ is a CA-discretization of X. Since a finite modal operator is completely additive the formula for $\mathbf{\Theta}$ follows from Lemma 2.3. □

[13] The convex sets in the pre-order $(\mathcal{X}, \leqslant)$ are the *locally closed* subsets in $\mathcal{X}$ as topological space.

[14] The expression $\mathbf{\Phi}^{+=} := \bigcup_{k\geqslant 0} \mathbf{\Phi}^k$ is the transitive reflexive closure of $\mathbf{\Phi}$, cf. App. A.1.

The advantage of using a discrete modal operator on $\mathsf{Set}(\mathfrak{X})$ is a refinement of the specialization pre-order on $\mathfrak{X}$ in terms of specialization relation given by:

$$(\xi, \xi') \in \phi \quad \text{if and only if} \quad \xi \in \mathbf{\Phi}\{\xi'\}. \tag{2.10}$$

Observe that ϕ need not be transitive and is not reflexive in general. Moreover, the transitive reflexive closure $\phi^{+=}$ is the specialization pre-order of $\mathbf{cl}_{\mathbf{\Phi}} = \mathbf{\Phi}^{+=}$ in (2.8), cf. Sect. 2.2.

REMARK 2.12. As for pre-orders the notion of down-set for a specialization relation is formulated as: $\mathcal{U} \subset \mathfrak{X}$ is a down-set for $\mathbf{\Phi}$ if $\xi' \in \mathcal{U}$ and $(\xi, \xi') \in \phi$, then $\xi' \in \mathcal{U}$. Note that $\mathsf{O}(\mathfrak{X}, \leqslant_{\mathbf{\Phi}}) = \mathsf{O}(\mathfrak{X}, \phi) = \mathsf{Fwdset}(\mathbf{\Phi})$.

REMARK 2.13. Modal algebra discretization using a derivative operator will be referred to as *DA-discretizations*. Binary relations coming from derivative operators will be called *weakly transitive*, or wK4, cf. [17]. If Γ satisfies the stronger sub-idempotency axiom in (K3), i.e. $\Gamma(\Gamma U) \subset \Gamma U$, we say that Γ is a *strong derivative operator*. This is often the case in dynamics in which instance the associated specialization relation is transitive (a K4-order).

2.4. Discretization maps

Let $(X, \mathscr{T})$ be a topological space. A *discretization map* on X is a *surjective* map[15]

$$\mathrm{disc}\colon X \longrightarrow\!\!\!\!\!\rightarrow \mathfrak{X}, \tag{2.11}$$

where $\mathfrak{X}$ is a finite set. Since unions and intersections are preserved under preimage, $\mathrm{disc}^{-1}\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(X)$, is an injective Boolean homomorphism, cf. Footn.'s 9 and 15. In this context, we say that disc^{-1} is an *evaluation map* and we use the notation: $|\mathcal{U}| := \mathrm{disc}^{-1}\mathcal{U}$ for $\mathcal{U} \in \mathsf{Set}(\mathfrak{X})$.

2.4.1. Topology consistent pre-orders. As pointed out above any topology on $\mathfrak{X}$ is equivalent to its specialization pre-order $(\mathfrak{X}, \leqslant)$. We say that $\leqslant$ is a $\mathscr{T}$-*consistent pre-order* on X with respect to disc if $\mathrm{disc}\colon (X, \mathscr{T}) \twoheadrightarrow (\mathfrak{X}, \leqslant)$ is continuous, which is equivalent to the condition that $\mathrm{cl}\,\mathrm{disc}^{-1}\mathcal{U} \subset \mathrm{disc}^{-1}\!\downarrow\!\mathcal{U}$, for all $\mathcal{U} \subset \mathfrak{X}$, i.e. $\mathrm{cl}|\mathcal{U}| \subset |\mathbf{cl}\,\mathcal{U}|$, where $\mathbf{cl}\,\mathcal{U} = \downarrow\!\mathcal{U}$. Consequently, when $\leqslant$ is $\mathscr{T}$-consistent, the triple $(\mathfrak{X}, \mathbf{cl}, |\cdot|)$ is a CA-discretization, and $\mathrm{disc}\colon (X, \mathscr{T}) \twoheadrightarrow (\mathfrak{X}, \leqslant)$ is a *continuous* discretization map. If disc is a continuous *open* map then $|\cdot| = \mathrm{disc}^{-1}$ is a homomorphism of closure algebras.[16] A specialization relation $\phi \subset \mathfrak{X} \times \mathfrak{X}$ is $\mathscr{T}$-consistent if the reflexive transitive closure is $\mathscr{T}$-consistent.

---

[15] As pointed in Footnote 9, in the context of discretization the maps are chosen to be surjective and are dual to injective closure algebra homomorphisms.

[16] By (2.8) $\xi \leqslant \xi'$ if and only if $\xi \in \mathbf{cl}\,\xi'$ which is equivalent to $\{\xi\} \subset \mathbf{cl}\,\xi'$. In the case that $|\cdot|$ is an injective homomorphism of closure algebras we obtain the equivalent statement $|\xi| \subset |\mathbf{cl}\,\xi'| = \mathrm{cl}|\xi'|$. Here we use the convention $\mathbf{cl}\{\xi\} = \mathbf{cl}\,\xi$.

Conversely, given a CA-discretization $(\mathfrak{X}, \mathbf{cl}, |\cdot|)$, we define $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ via

$$\mathrm{disc}(x) = \xi \text{ where } x \in |\xi|, \tag{2.12}$$

which is a well-defined map since $\bigcup_\xi |\xi| = X$ and $|\xi|$ and $|\xi'|$ are mutually disjoint for all $\xi \neq \xi'$, cf. Rem. 2.5. By Birkhoff duality the injectivity of $|\cdot|$ implies the surjectivity of disc, cf. Thm. B.2 and [15, Thm. 5.19]. Moreover, since $\mathrm{cl}|\mathcal{U}| \subset |\mathbf{cl}\,\mathcal{U}|$, $|\cdot| = \mathrm{disc}^{-1}$, the map disc is a continuous map, thus $\leqslant$ is $\mathscr{T}$-consistent. For a discretization of a subset X of the plane with the associated face partial order $\leqslant$ in Fig. 1.1[left 1 and 2] the pre-order $(\mathfrak{X}, \leqslant)$ is a discretization of the topology $\mathscr{T}$ of X. The map $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ assigns a vertex, edge or square to any point in X. For closure algebras, disc is induced by sending a point x to the cell in which it is contained, as in Equation (2.12). We can summarize these considerations as follows:

PROPOSITION 2.14. *A surjective, continuous discretization map* $\mathrm{disc}\colon (X, \mathscr{T}) \longrightarrow\!\!\!\!\!\rightarrow (\mathfrak{X}, \leqslant)$ *is equivalent to a CA-discretization* $(\mathfrak{X}, \mathbf{cl}, |\cdot|)$ *with* $(\mathsf{Set}(\mathfrak{X}), \mathbf{cl})$ *dual*[17] *to* $(\mathfrak{X}, \leqslant)$ *and* $|\cdot| = \mathrm{disc}^{-1}$.

A pre-order $\leqslant$ is a *$\mathscr{T}$-co-consistent* with respect to disc if $\mathrm{disc}\colon X \to (\mathfrak{X}, \geqslant)$ is continuous, where $\geqslant$ is the opposite pre-order. This is equivalent to the condition $\mathrm{cl}\,\mathrm{disc}^{-1}\mathcal{U} \subset \mathrm{disc}^{-1}\bar{\mathbf{cl}}\,\mathcal{U}$, for all $\mathcal{U} \subset \mathfrak{X}$, where $\bar{\mathbf{cl}}\,\mathcal{U} := \mathbf{star}\,\mathcal{U}$ is the *conjugate closure operator*, cf. [51]. In terms of realization this reads $\mathrm{cl}|\mathcal{U}| \subset |\,\mathbf{star}\,\mathcal{U}|$. If $\mathcal{U}$ is $\bar{\mathbf{cl}}$-closed, i.e. $\bar{\mathbf{cl}}\,\mathcal{U} = \mathbf{star}\,\mathcal{U} = \mathcal{U}$ which implies that $\mathcal{U}$ is open in $(\mathfrak{X}, \leqslant)$, then the $\mathscr{T}$-co-consistency implies that $|\mathcal{U}|$ is a closed set. Indeed, $\mathrm{cl}|\mathcal{U}| \subset |\,\mathbf{star}\,\mathcal{U}| = |\mathcal{U}| \subset \mathrm{cl}|\mathcal{U}|$ and thus $\mathrm{cl}|\mathcal{U}| = |\mathcal{U}|$. Moreover, the closed sets $\mathcal{U} \in \mathsf{O}(\mathfrak{X}, \leqslant)$ for a $\mathscr{T}$-co-consistent pre-order are open under realization. If $\mathcal{U} \in \mathsf{O}(\mathfrak{X}, \leqslant)$ then $\mathbf{cl}\,\mathcal{U} = \mathcal{U}$ and therefore $\mathcal{U}^c$ is open which implies that $\mathbf{star}\,\mathcal{U}^c = \mathcal{U}^c$. By the previous $|\mathcal{U}^c| = |\mathcal{U}|^c$ is closed and thus $|\mathcal{U}|$ is open.

REMARK 2.15. If we allow the evaluation map $|\cdot|\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(X)$ to be an arbitrary homomorphism then the map disc defined in (2.12) is still valid which allows us to treat the theory of discretization with arbitrary, not necessarily surjective, continuous maps $\mathrm{disc}\colon X \to \mathfrak{X}$. In this paper we restrict to surjective discretization maps unless stated otherwise.

REMARK 2.16. For a discretization map $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$, there is always a $\mathscr{T}$-consistent pre-order. Namely, the trivial, or indiscrete topology on $\mathfrak{X}$: $\mathrm{cl}\,\varnothing = \varnothing$ and $\mathrm{cl}\,\mathcal{U} = \mathfrak{X}$ for all $\mathcal{U} \neq \varnothing$, i.e. $\leqslant$ is an equivalence: $\xi \leqslant \xi'$ and $\xi' \leqslant \xi$ for all $\xi, \xi' \in \mathfrak{X}$.

2.4.2. Filtering and grading. If $\leqslant$ is a $\mathscr{T}$-consistent pre-order for a discretization map disc, then by (2.8) the down-sets $\mathcal{U}$ for $\leqslant$ correspond to the closed sets in $\mathfrak{X}$ and therefore by the continuity of disc we have that $\mathrm{disc}^{-1}\mathcal{U} \in \mathscr{C}(X, \mathscr{T})$, where

[17] In terms of Boolean algebras with operators, cf. App. B.3 the pre-order is the dual to the closure algebra and vice versa. This duality can also be understood in terms of (co)-Heyting algebras.

$\mathscr{C}(X) = \mathscr{C}(X, \mathscr{T})$[18] denotes the closed sets in X. This yields the lattice homomorphism

$$\operatorname{disc}^{-1}\colon \mathsf{O}(\mathfrak{X}, \leqslant) \longrightarrow \mathscr{C}(X) \xrightarrow{\subset} \mathsf{Set}(X). \tag{2.13}$$

In this setting we refer to disc^{-1} as an $\mathsf{O}(\mathfrak{X}/_\sim)$-*filtering* on X and use the filtering notation: $F_{\mathcal{U}}X := \operatorname{disc}^{-1}\mathcal{U} = |\mathcal{U}|$. Similarly, a subset $\mathcal{U} \subset \mathfrak{X}$ is *open* if it is an up-set of $\mathfrak{X}$ and the image under disc^{-1} are open sets in X. Dual to the $\mathsf{O}(\mathfrak{X}/_\sim)$-filtering is the *grading* $X = \bigcup_{\xi \in [\xi]} G_{[\xi]}X$ with the property that $G_{[\xi]}X \neq \varnothing$ for all $\xi \in \mathfrak{X}$, cf. App. C.1. The latter is an $\mathfrak{X}/_\sim$-grading given by $G_{[\xi]}X \xmapsto{\operatorname{grd}} [\xi]$, where $\{G_{[\xi]}X \mid [\xi] \in (\mathfrak{X}/_\sim, \leqslant)\}$ is an ordered tessellation. The following scheme shows the duality between CA-discretizations and continuous discretization maps, and between filterings and gradings, cf. App. C.1:

$$\begin{tikzcd}
\begin{array}{c}(\mathfrak{X}, |\cdot|, \mathrm{cl})\\ \text{CA-discretization}\end{array} \arrow[r] \arrow[d, leftrightarrow] & \begin{array}{c}|\cdot|\colon \mathsf{O}(\mathfrak{X}, \leqslant) \to \mathsf{Set}(X)\\ \text{filtering}\end{array} \arrow[d, leftrightarrow] \\
\begin{array}{c}\operatorname{disc}\colon (X, \mathscr{T}) \twoheadrightarrow (\mathfrak{X}, \leqslant)\\ \text{continuous discretization}\end{array} \arrow[r] & \begin{array}{c}X = \bigcup_{[\xi]} G_{[\xi]}X\\ \text{grading}\end{array}
\end{tikzcd}$$

where $G_{[\xi]}X = \operatorname{disc}^{-1}[\xi]$. In general a grading on a topological space yields a discretization which is not necessarily continuous.[19] This implies that the above diagram do not necessarily point in the opposite direction. Finally we define a class of discretization maps which are favorable for using homology theories.

DEFINITION 2.17. A discretization map $\operatorname{disc}\colon X \twoheadrightarrow \mathfrak{X}$ is *natural* if it is continuous and the associated filtering $\operatorname{disc}^{-1}\colon \mathsf{O}(\mathfrak{X}, \leqslant) \to \mathsf{Set}(X)$ consists of mutually good pairs.[20]

2.5. Bi-topological spaces and discretization

A triple $(X, \mathscr{T}, \mathscr{T}')$ is called a *bi-topological space* if the factors $(X, \mathscr{T})$ and $(X, \mathscr{T}')$ are well-defined topological spaces. The associated closure algebra for $(X, \mathscr{T}, \mathscr{T}')$ is the Boolean algebra with operators $(\mathsf{Set}(X), \mathrm{cl}, \mathrm{cl}')$, where cl and cl' are the closure operators for $\mathscr{T}$ and $\mathscr{T}'$ respectively and is referred to as *bi-closure algebra* for $(X, \mathscr{T}, \mathscr{T}')$. A subset $U \subset X$ is a $(\mathscr{T}, \mathscr{T}')$-*pairwise clopen set* for X if U is closed in $\mathscr{T}$ and open in $\mathscr{T}'$. We denote the set of $(\mathscr{T}, \mathscr{T}')$-pairwise clopen sets

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[18] If there is no ambiguity about the topology we write $\mathscr{C}(X)$ for short. The same applies to the open sets $\mathscr{O}(X)$.

[19] Let $X = \bigcup_{p \in \mathsf{P}} G_pX$ be a P-graded decomposition of X. Then, the map $\operatorname{grd}\colon X \to \mathsf{P}$, defined by $\operatorname{grd}(x) = \operatorname{grd}(G_pX) = p$ for all $x \in G_pX$, is a discretization map in the sense of Rmk. 2.15. By restricting to the range one obtains a surjective discretization map, cf. App. C.1.

[20] Recall that a pair (X, A), $A \subset X$ closed, is a *good pair* if A is a deformation retract of a neighborhood in X, q.v. [34, Thm. 2.13].

in a bi-topological space $(X, \mathscr{T}, \mathscr{T}')$ by $\mathscr{C}\mathscr{O}(X)$. Similarly, we can define $(\mathscr{T}', \mathscr{T})$-pairwise clopen sets which are open in $\mathscr{T}$ and closed in $\mathscr{T}'$ and are denoted by $\mathscr{C}\mathscr{O}^*(X)$.

The next step is to consider discretization for bi-topological spaces. A discretization map $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ is a p-continuous map[21] between bi-topological spaces if there exists pre-orders $(\mathfrak{X}, \leqslant)$ and $(\mathfrak{X}, \leqslant')$ that are $\mathscr{T}$-consistent and $\mathscr{T}'$-consistent respectively. We write $\mathrm{disc}\colon (X, \mathscr{T}, \mathscr{T}') \twoheadrightarrow (\mathfrak{X}, \leqslant, \leqslant')$. The associated *bi-topological CA-discretization* is denoted by $(\mathfrak{X}, \mathbf{cl}, \mathbf{cl}', |\cdot|)$ where $\mathbf{cl}$ and $\mathbf{cl}'$ are the associated closure operators. Let $(\mathfrak{X}, \mathbf{cl}, \mathbf{cl}', |\cdot|)$ be a bi-topological CA-discretization for $(X, \mathscr{T}, \mathscr{T}')$. This is equivalent to choosing a discretization map $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ and pre-orders $(\mathfrak{X}, \leqslant)$ and $(\mathfrak{X}, \leqslant')$ such that disc is continuous with respect to both $\mathscr{T}$ and $\mathscr{T}'$. Since the pre-orders $\leqslant$ and $\leqslant'$ represent Alexandrov topologies we can coarsen the finite topologies using both up-sets and down-sets.

DEFINITION 2.18. Let $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ be a discretization map. An *antagonistic pre-order* for $(X, \mathscr{T}, \mathscr{T}')$ is a pre-order $(\mathfrak{X}, \leqslant^\dagger)$ such that

(i) $\leqslant^\dagger$ is $\mathscr{T}$-consistent with respect to disc;
(ii) $\leqslant^\dagger$ is $\mathscr{T}'$-co-consistent with respect to disc.

These conditions translate as

$$\mathrm{cl}|\mathcal{U}| \subset |\mathbf{cl}^\dagger\mathcal{U}|, \quad \mathrm{cl}'|\mathcal{U}| \subset |\,\mathbf{star}^\dagger\,\mathcal{U}|, \quad \forall \mathcal{U} \subset \mathfrak{X}, \tag{2.14}$$

where $\mathbf{star}^\dagger = \bar{\mathbf{c}}\mathbf{l}^\dagger$, the conjugate closure operator, cf. Sect. 2.4.1 and [38]. The triple $(\mathfrak{X}, \mathbf{cl}^\dagger, |\cdot|)$ is called an *antagonistic CA-discretization* for $(X, \mathscr{T}, \mathscr{T}')$.

REMARK 2.19. An equivalent way to say that $(\mathfrak{X}, \leqslant^\dagger)$ is an antagonistic pre-order is that both

$$\mathrm{disc}\colon (X, \mathscr{T}) \longrightarrow\!\!\!\!\!\rightarrow (\mathfrak{X}, \leqslant^\dagger), \quad \text{and} \quad \mathrm{disc}\colon (X, \mathscr{T}') \longrightarrow\!\!\!\!\!\rightarrow (\mathfrak{X}, \geqslant^\dagger),$$

are continuous.

REMARK 2.20. We can use the pairwise clopen sets in $(X, \mathscr{T}, \mathscr{T}')$ as a base (of closed sets) for the topology $\mathscr{T}^\dagger$. Closed sets in $(\mathfrak{X}, \leqslant^\dagger)$ are pairwise clopen sets in $(X, \mathscr{T}, \mathscr{T}')$, which yields the continuous discretization map $\mathrm{disc}\colon (X, \mathscr{T}^\dagger) \twoheadrightarrow (\mathfrak{X}, \leqslant^\dagger)$. We say that $\mathscr{T}^\dagger$ is the *antagonistic topology* with respect to the pair $(\mathscr{T}, \mathscr{T}')$. Since closed sets in $(X, \mathscr{T}^\dagger)$ are not necessarily pairwise clopen it is preferable to use the concept of pairwise clopen sets in the bi-topological space $(X, \mathscr{T}, \mathscr{T}')$.

REMARK 2.21. Antagonistic pre-orders can also be defined by reversing the role of $\mathscr{T}$ and $\mathscr{T}'$.

Antagonistic pre-orders yield discrete topologies on $\mathfrak{X}$ since such topologies are necessarily Alexandrov. One cannot play the same game with $\mathscr{T}$ and $\mathscr{T}'$ on

[21]J.C. Kelly refers to such map that are continuous with respect to both topologies as *p-continuous*, or *pairwise continuous* maps, cf. [45].

X since topologies are not Alexandrov in general. For this reason one chooses the formalism of bi-topological spaces and anatagonistic pre-orders.

Let $(\mathfrak{X}, \leqslant^\dagger)$ be an antagonistic pre-order as in Definition 2.18. Closed sets $\mathcal{U}$ in $(\mathfrak{X}, \leqslant^\dagger)$ are pairwise clopen sets $|\mathcal{U}| \in \mathscr{C}\mathscr{O}(X)$, cf. Sect. 2.4.1. Conversely, if $(\mathfrak{X}, \mathbf{cl}, \mathbf{cl}', |\cdot|)$ is a bi-topological CA discretization for $(X, \mathscr{T}, \mathscr{T}')$ then the lattice embedding:

$$\iota\colon \mathsf{O}(\mathsf{P}) \rightarrowtail \mathsf{O}(\mathfrak{X}, \leqslant) \cap \mathsf{U}(\mathfrak{X}, \leqslant') =: \mathscr{C}\mathscr{O}(\mathfrak{X}), \tag{2.15}$$

yields an antagonistic pre-order for $(X, \mathscr{T}, \mathscr{T}')$. Indeed, if we use Birkhoff duality to dualize the homomorphism $\iota\colon \mathsf{O}(\mathsf{P}) \rightarrowtail \mathsf{Set}(\mathfrak{X})$, q.v. Thm. B.2 and Rem. B.3, we obtain the order-preserving surjection

$$\begin{aligned} \pi\colon \mathfrak{X} \twoheadrightarrow \mathsf{P}, \quad \xi \mapsto \pi(\xi) &:= \min\{p \in \mathsf{P} \mid \xi \in \iota(\downarrow p)\} \\ &= \max\Big\{\min\{\mathcal{U} \in \mathsf{J}(\mathsf{O}(\mathsf{P})) \mid \xi \in \mathcal{U}\}\Big\}, \end{aligned} \tag{2.16}$$

where $\mathsf{J}\big(\mathsf{O}(\mathsf{P})\big)$ is the poset of join-irreducible elements in $\mathsf{O}(\mathsf{P})$. By construction a pre-order $(\mathfrak{X}, \leqslant^\dagger)$ is defined by

$$\xi \leqslant^\dagger \xi' \quad \text{if and only if} \quad \pi(\xi) \leqslant \pi(\xi'),$$

is an antagonistic pre-order with $\mathsf{O}(\mathfrak{X}, \leqslant^\dagger) \cong \mathsf{O}(\mathsf{P})$. In this case we say that $\leqslant^\dagger$ is an *antagonistic coarsening* of $(\mathfrak{X}, \mathbf{cl}, \mathbf{cl}', |\cdot|)$. The lattice embedding $\mathsf{O}(\mathfrak{X}, \leqslant^\dagger) \rightarrowtail \mathsf{Set}(\mathfrak{X})$ is dual to the identity map

$$\mathrm{id}\colon \mathfrak{X} \to (\mathfrak{X}, \leqslant^\dagger). \tag{2.17}$$

THEOREM 2.22. *A pre-order $\leqslant^\dagger$ is an antagonistic pre-order for* $\mathrm{disc}\colon (X, \mathscr{T}, \mathscr{T}') \to (\mathfrak{X}, \leqslant^\dagger)$ *if and only if there exists a bi-topological CA-discretization* $(\mathfrak{X}, \mathbf{cl}, \mathbf{cl}', |\cdot|)$ *such that* $\mathsf{O}(\mathfrak{X}, \leqslant^\dagger)$ *is given by* (2.15)*, i.e. an antagonistic pre-order is equivalent to an antagonistic coarsening.*

PROOF. One direction is given be the construction in (2.15). It remains to show that an antagonistic pre-order satisfying Definition 2.18(i)-(ii) is an antagonistic coarsening. If we define the discrete closure operators $\mathbf{cl} = \mathbf{cl}^\dagger$ and $\mathbf{cl}' = \mathbf{star}^\dagger$, then $\mathsf{O}(\mathfrak{X}, \leqslant) = \mathsf{O}(\mathfrak{X}, \leqslant^\dagger)$ and $\mathsf{U}(\mathfrak{X}, \leqslant') = \mathsf{O}(\mathfrak{X}, \leqslant^\dagger)$, which proves the theorem. □

In practical situations, given a bi-topological CA-discretization $(\mathfrak{X}, \leqslant, \leqslant')$, we can choose $\mathsf{O}(\mathfrak{X}, \leqslant^\dagger) = \mathscr{C}\mathscr{O}(\mathfrak{X})$. Let $\mathsf{J}(\mathsf{O}(\mathfrak{X}, \leqslant^\dagger))$ be the poset of join-irreducible elements in $\mathscr{C}\mathscr{O}(\mathfrak{X})$. From the results in [44, 40] consider a representation $(\mathsf{SC}, \leqslant)$ of $\mathsf{J}(\mathsf{O}(\mathfrak{X}, \leqslant^\dagger))$ defined by

$$\mathsf{SC} := \big\{\mathcal{S} = \mathcal{U} \smallsetminus \mathcal{U}^\blacktriangleleft \mid \mathcal{U} \in \mathsf{J}(\mathsf{O}(\mathfrak{X}, \leqslant^\dagger))\big\}, \tag{2.18}$$

with $\mathcal{S} \leqslant \mathcal{S}'$ if and only if $\mathcal{U} \subset \mathcal{U}'$, with $\mathcal{U}, \mathcal{U}' \in \mathsf{J}(\mathsf{O}(\mathfrak{X}, \leqslant^\dagger))$ uniquely determined by $\mathcal{S} = \mathcal{U} \smallsetminus \mathcal{U}^\blacktriangleleft$ and $\mathcal{S}' = \mathcal{U}' \smallsetminus \mathcal{U}'^\blacktriangleleft$.[22] If we regard the pre-order $(\mathfrak{X}, \leqslant^\dagger)$ as a directed graph then the sets $\mathcal{S} \in \mathsf{SC}$ correspond to the *strongly connected components* of the directed graph which is the motivation for the abbreviation SC, cf.

[22] For every $\mathcal{U} \in \mathsf{J}(\mathsf{O}(\mathfrak{X}, \leqslant^\dagger))$ there exists a unique immediate predecessor $\mathcal{U}^\blacktriangleleft \in \mathsf{O}(\mathfrak{X}, \leqslant^\dagger)$.

Rem. 2.24. The posets $(\mathsf{SC},\leqslant)$ and $\big(\mathsf{J}(\mathsf{O}(\mathscr{X},\leqslant^\dagger)),\subset\big)$ are isomorphic by construction and the elements in SC correspond to the partial equivalence classes in $(\mathscr{X},\leqslant^\dagger)$, cf. Rem. 2.10, which yields the (order-preserving) projection $\mathrm{dyn}\colon(\mathscr{X},\leqslant^\dagger)\twoheadrightarrow(\mathsf{SC},\leqslant) = \mathscr{X}/_\sim$.[23] The latter may be regarded as a *finite* discretization of $\mathscr{X}$. The embedding $\mathsf{O}(\mathscr{X},\leqslant^\dagger)\xrightarrow{\subset}\mathsf{O}(\mathscr{X},\leqslant)$ implies that dyn is also an order-preserving map $\mathrm{dyn}\colon(\mathscr{X},\leqslant)\twoheadrightarrow(\mathsf{SC},\leqslant)$ which factors as $(\mathscr{X},\leqslant)\xrightarrow{\mathrm{id}}(\mathscr{X},\leqslant^\dagger)\xrightarrow{\mathrm{dyn}}(\mathsf{SC},\leqslant)$. Indeed, a downset in $(\mathscr{X},\leqslant^\dagger)$ is a downset in $(\mathscr{X},\leqslant)$ and therefore $\leqslant\subset\leqslant^\dagger$ as pre-orders. By the same token we have that dyn is order-reversing with respect to $\leqslant'$ which follows from the embedding $\mathsf{O}(\mathscr{X},\leqslant^\dagger)\xrightarrow{\subset}\mathsf{U}(\mathscr{X},\leqslant')$ and therefore $\geqslant'\subset\leqslant^\dagger$ as pre-orders. Summarizing, the maps dyn are order-preserving and order-reversing respectively:

(2.19)
$$
\begin{tikzcd}
 & (\mathscr{X},\leqslant) \arrow[d, "\mathrm{id}"] \arrow[dr, "\mathrm{dyn}"] \arrow[drr, bend left, "\mathrm{part}"] & & \\
(X,\mathscr{T},\mathscr{T}') \arrow[ur, "\mathrm{disc}"] \arrow[r, bend left, "\mathrm{disc}"] \arrow[r, dashed, bend right, "\mathrm{disc}"'] \arrow[dr, "\mathrm{disc}"'] & (\mathscr{X},\leqslant^\dagger) \arrow[r, "\mathrm{dyn}"] & \mathsf{SC} \arrow[r, two heads, "\pi"] & \mathsf{P} \\
 & (\mathscr{X},\leqslant') \arrow[u, dashed, "\mathrm{id}"] \arrow[ur, dashed, "\mathrm{dyn}"'] \arrow[urr, dashed, bend right, "\mathrm{part}"'] & &
\end{tikzcd}
$$

The map dyn is again a continuous discretization and the composition

(2.20)
$$
\begin{tikzcd}
X \arrow[r, two heads, "\mathrm{disc}"] \arrow[rr, two heads, bend right, "\mathrm{tile}"'] & \mathscr{X} \arrow[r, two heads, "\mathrm{dyn}"] & \mathsf{SC}
\end{tikzcd}
$$

denoted by tile, may be regarded as a continuous map $\mathrm{tile}\colon(X,\mathscr{T})\to(\mathsf{SC},\leqslant)$ and as a continuous map $(X,\mathscr{T}')\to(\mathsf{SC},\geqslant)$ by factoring through $(\mathscr{X},\leqslant^\dagger)$ and $(\mathscr{X},\geqslant^\dagger)$ respectively, and which is equivalently obtained by factoring through $(\mathscr{X},\leqslant)$ by factoring through $(\mathscr{X},\leqslant')$ respectively. Every antagonistic pre-order for $(X,\mathscr{T},\mathscr{T}')$ defines a grading $G_{\mathcal{S}}X\xmapsto{\mathrm{tile}}\mathcal{S}$ of X given by

$$X=\bigcup_{\mathcal{S}\in\mathsf{SC}}G_{\mathcal{S}}X,\quad G_{\mathcal{S}}X=\mathrm{tile}^{-1}\mathcal{S},$$

which is called an *antagonistic tessellation* of X. We apply these bi-topological discretization techniques in the next two chapters in the context of discretizing semi-flows.

REMARK 2.23. An antagonistic pre-order $(\mathscr{X},\leqslant^\dagger)$ satisfies $\leqslant\subset\leqslant^\dagger$ and $\geqslant'\subset\leqslant^\dagger$ as pre-orders and thus $\leqslant\,\vee\,\geqslant'\subset\leqslant^\dagger$ as pre-orders. The 'vee' on the pre-order is the transitive closure of the union. The case $\mathsf{O}(\mathscr{X},\leqslant^\dagger)=\mathscr{C}\mathscr{O}(\mathscr{X})$ corresponds to $\leqslant\,\vee\,\geqslant'=\leqslant^\dagger$.

[23]We use the terminology dyn as it is used later on in the setting of the block-flow topology .

REMARK 2.24. There are various (equivalent) perspectives for presenting binary relations on $\mathcal{X}$:

(a) as a binary relation, i.e. $\phi \subset \mathcal{X} \times \mathcal{X}$;
(b) as a directed graph (digraph) ϕ with vertices $\mathcal{X}$ and edge set
$$\{\xi \to \xi' \mid (\xi, \xi') \in \phi^{-1}\};$$
(c) as modal operator $\mathbf{\Phi} = \phi^{-1}\colon \mathsf{Set}(\mathcal{X}) \to \mathsf{Set}(\mathcal{X})$.

We alternate between these perspectives, using whichever is conceptually most convenient. As digraph the strongly connected components are found via (A.3).

2.6. Examples and further extensions

In this section we discuss examples of discretization of topology starting with regular closed sets. The latter will be applied in the setting of CW-decompositions. We start with outlining regular closed sets.

2.6.1. Regular closed sets. Of particular importance in this text are the regular closed sets which play a central role in the construction of Morse tessellations. A subset $U \subset X$ is *regular closed* if $\mathrm{cl}\,\mathrm{int}\, U = U$. The set of regular closed sets in any topological space X is denoted by $\mathscr{R}(X, \mathscr{T})$ and the latter forms a complete Boolean algebra with unary operation $U^{\#} = \mathrm{cl}\, U^c$ and binary operations $U \vee U' = U \cup U'$ and $U \wedge U' = \mathrm{cl}\,\mathrm{int}(U \cap U')$, cf. [68, Sect. 2.3]. By the same token we can define the regular closed sets in a CA-discretization $(\mathcal{X}, \mathrm{cl}, |\cdot|)$ and we denote regular closed sets in $\mathcal{X}$ by $\mathscr{R}(\mathcal{X})$.[24] Regular closed subsets can be obtained from closed subsets $\mathsf{O}(\mathcal{X}, \leqslant)$.

PROPOSITION 2.25 (cf. [44], Lem. 22). *The map* $\mathbf{cl}\,\mathbf{int}\colon \mathsf{O}(\mathcal{X}, \leqslant) \to \mathscr{R}(\mathcal{X})$, *defined by* $\mathcal{U} \mapsto \mathbf{cl}\,\mathbf{int}\,\mathcal{U}$, *is a surjective lattice homomorphism.*[25]

The atoms in $\mathsf{Set}(\mathcal{X})$ are given by the set $\mathcal{X}$. Since $\mathscr{R}(\mathcal{X})$ is a finite Boolean algebra it is a power set on a set of atoms. Define the maximal elements in $\mathcal{X}$ with respect to $\leqslant$ by $\mathcal{X}^\top$, i.e. $\xi \in \mathcal{X}^\top$ if and only if $\mathbf{star}\,\xi = \xi$. Such elements are called *top cells* and form an anti-chain in $(\mathcal{X}, \leqslant)$.

PROPOSITION 2.26. *The atoms of* $\mathscr{R}(\mathcal{X})$ *are given by the set* $\{\mathbf{cl}\,\xi \mid \xi \in \mathcal{X}^\top\}$. *Moreover,* $\mathbf{cl}\colon \mathsf{Set}(\mathcal{X}^\top) \to \mathscr{R}(\mathcal{X})$ *is an isomorphism with inverse* $\mathcal{U} \mapsto \mathcal{U} \cap \mathcal{X}^\top$.

PROOF. We start with the observation that $\xi \in \mathcal{X}^\top$ represents an open subset. Indeed, $\mathbf{star}\,\xi = \xi$ and thus $\xi = \mathbf{int}\,\xi$. Then, $\mathbf{cl}\,\xi = \mathbf{cl}\,\mathbf{int}\,\xi$ is a regular closed set since $\mathbf{cl}\,\mathbf{int}\,\mathbf{cl}\,\mathbf{int}\,\xi = \mathbf{cl}\,\mathbf{int}\,\xi$. Suppose $\xi \notin \mathcal{X}^\top$, i.e. ξ is not maximal in $(\mathcal{X}, \leqslant)$. Since $\xi = \{\xi\}$ is a singleton set and since $\mathbf{int}\{\xi\} \subset \{\xi\}$ we conclude that $\mathbf{int}\,\xi = \xi$, or $\mathbf{int}\,\xi = \varnothing$. Suppose the former holds. Then, since ξ is not maximal, $\{\xi\} \subsetneq \mathbf{star}\,\xi =$

[24] In this text we mainly consider regular closed sets with repect to one topology $\mathscr{T}$. Therefore the notation $\mathscr{R}(X)$ and $\mathscr{R}(\mathcal{X})$ does not cause any ambiguities.

[25] This result holds for any topological space X.

$\{\xi\}$, a contradiction. Therefore, $\mathbf{int}\,\xi = \varnothing$. Let $\mathcal{U} = \bigcup\{\xi\}$ be a regular closed set. Then, by invoking Proposition 2.25 we have

$$\mathcal{U} = \mathbf{cl}\,\mathbf{int}\,\mathcal{U} = \mathbf{cl}\,\mathbf{int}\Big(\bigcup\{\xi\}\Big) = \bigcup \mathbf{cl}\,\mathbf{int}\,\xi = \bigcup\Big\{\mathbf{cl}\,\xi \mid \xi \in \mathfrak{X}^\top\Big\},$$

which proves that every regular closed set is union of elements $\mathbf{cl}\,\xi$, for $\xi \in \mathfrak{X}^\top$. The latter form an anti-chain in $\mathscr{R}(\mathfrak{X})$, i.e., again invoking Proposition 2.25,

$$\mathbf{cl}\,\xi \wedge \mathbf{cl}\,\xi' = \mathbf{cl}\,\mathbf{int}\,\xi \wedge \mathbf{cl}\,\mathbf{int}\,\xi' = \mathbf{cl}\,\mathbf{int}\big(\xi \cap \xi'\big) = \varnothing, \tag{2.21}$$

which proves that $\mathbf{cl}\,\xi$, for $\xi \in \mathfrak{X}^\top$, are atoms. The map $\mathbf{cl}\colon \mathsf{Set}(\mathfrak{X}^\top) \to \mathscr{R}(\mathfrak{X})$ preserves union. Consider $\mathbf{cl}\,\mathcal{U} \wedge \mathbf{cl}\,\mathcal{U}'$. By (2.21) we have

$$\mathbf{cl}\,\mathcal{U} \wedge \mathbf{cl}\,\mathcal{U}' = \bigcup \mathbf{cl}\,\xi \wedge \bigcup \mathbf{cl}\,\xi' = \bigcup \mathbf{cl}\,\xi'' = \mathbf{cl}\Big(\bigcup\{\xi''\}\Big),$$

where $\bigcup\{\xi''\} = \mathcal{U} \cap \mathcal{U}'$, which proves that $\mathbf{cl}\colon \mathsf{Set}(\mathfrak{X}^\top) \to \mathscr{R}(\mathfrak{X})$ is a homomorphism. Any $\mathcal{U} \in \mathscr{R}(\mathfrak{X})$ is uniquely represented as $\mathcal{U} = \bigcup \mathbf{cl}\,\xi = \mathbf{cl}\big(\bigcup\{\xi\}\big)$, $\xi \in \mathfrak{X}^\top$ which shows that $\mathcal{U} \cap \mathfrak{X}^\top$ yields the unique set of generating cells $\xi \in \mathfrak{X}^\top$. $\square$

Regular closed sets in $\mathfrak{X}$ do not necessarily yield regular closed sets in X under evaluation. However, if $(\mathfrak{X}, \mathbf{cl}, |\cdot|)$ is *Boolean*, cf. Defn. 2.9, then we have the following correspondence: $|\mathcal{U}| = |\mathbf{cl}\,\mathbf{int}\,\mathcal{U}| = \mathrm{cl}|\,\mathbf{int}\,\mathcal{U}| = \mathrm{cl}\,\mathrm{int}\,|\mathcal{U}|$[26] and thus a subset $\mathcal{U} \subset \mathfrak{X}$ is regular closed in $\mathfrak{X}$ if and only if $|\mathcal{U}| \subset X$ is regular closed in X. This way the image of $|\cdot|\colon \mathscr{R}(\mathfrak{X}) \to \mathscr{R}(X)$ yields the finite Boolean subalgebra $\mathscr{R}_0(X)$ contained in $\mathscr{R}(X)$. The subalgebra $\mathscr{R}_0(X)$ is generated by the set of atoms $\mathsf{J}\big(\mathscr{R}_0(X)\big) := \big\{\mathrm{cl}|\xi| \mid \xi \in \mathfrak{X}^\top\big\}$.[27]

PROPOSITION 2.27. *Let $(\mathfrak{X}, \mathbf{cl}, |\cdot|)$ be a Boolean CA-discretization for X. Then, the map*

$$\|\cdot\|\colon \mathsf{Set}(\mathfrak{X}^\top) \to \mathscr{R}_0(X), \quad \xi \mapsto \|\xi\| := \mathbf{cl}|\xi|,$$[28]

is a lattice isomorphism and thus a Boolean isomorphism.

Conversely, a finite sub-algebra $\mathscr{R}_0(X) \subset \mathscr{R}(X)$ yields a finite closure algebra $\big(\mathsf{Set}(\mathfrak{X}), \mathbf{cl}\big)$ as follows. Define $[\mathscr{R}_0(X)]$ as the smallest sub-algebra in $\mathsf{Set}(X)$ containing $\mathscr{R}_0(X)$. Denote the set of atoms by $\mathsf{J}\big([\mathscr{R}_0(X)]\big) := \big\{|\xi| \mid \xi \in \mathfrak{X}\big\}$ for some finite set $\mathfrak{X}$. This defines the pre-order $(\mathfrak{X}, \leqslant)$ via the relation: $\xi \leqslant \xi'$ if and only if $|\xi| \subset \mathbf{cl}|\xi'|$. Via the pre-order we obtain a closure operator on $\mathsf{Set}(\mathfrak{X})$ via (2.8).

PROPOSITION 2.28. *Let $\mathscr{R}_0(X) \subset \mathscr{R}(X)$ be a finite sub-algebra. Then, $[\mathscr{R}_0(X)] \subset \mathsf{Set}(X)$ defines a unique finite closure algebra $\big(\mathsf{Set}(\mathfrak{X}), \mathbf{cl}\big)$, where the $\mathbf{cl}$ is defined by (2.8).*

The above statement can be rephrased as: a finite sub-algebra $\mathscr{R}_0(X) \subset \mathscr{R}(X)$ induces a unique Boolean CA-discretization $(\mathfrak{X}, \mathbf{cl}, |\cdot|)$ for X, where $|\cdot|$ an injective homomorphism of closure algebras.

[26] If $(\mathfrak{X}, \mathbf{cl}, |\cdot|)$ is Boolean then $\mathbf{int}\,|\mathcal{U}| = |\,\mathbf{int}\,\mathcal{U}|$ follows from the relation for closure.

[27] The join-irreducible element in a finite Boolean algebra and the atoms that generate the Boolean algebra.

[28] The notation $\|\cdot\|$ is called *closed realization*.

2.6.2. CW-decomposition maps. Consider a discretization $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ without indicating a specific pre-order on $\mathfrak{X}$ for now. Let B^q and $\bar{B}^q$ denote the open and closed unit balls in $\mathbb{R}^q$ respectively (where B^0 and $\bar{B}^0$ denote the one point space). We say that ξ is an q-*cell* if $|\xi|$ is homeomorphic to an open ball B^q. The integer q is called the *dimension* of ξ and is denoted $\dim \xi$. Suppose disc has the property that every ξ is an q-cell for some q. Given such a discretization map assigning dimension to a cell is an order-preserving map

$$\dim\colon \mathfrak{X} \to (\mathbb{N}, \leqslant),$$

when $\mathfrak{X}$ is regarded as anti-chain. Note therefore that the anti-chain $\mathfrak{X}$ is a naturally graded set with respect to dim, i.e., $\mathfrak{X} = \bigcup_{q\in\mathbb{N}} G_q\mathfrak{X}$, where $G_q\mathfrak{X} = \dim^{-1} q$ is the set of q-cells. The latter also yields the filtering

$$\downarrow q \mapsto F_{\downarrow q}\mathfrak{X}, \quad \downarrow q \in \mathsf{O}(\mathbb{N}),$$

where, by Birkhoff duality, $F_{\downarrow q}\mathfrak{X} = \dim^{-1}\downarrow q$, $\downarrow q = \{0, 1, \cdots, q\}$ and $q \in \mathbb{N}$. The composition $X \xrightarrow{\mathrm{disc}} \mathfrak{X} \xrightarrow{\dim} \mathbb{N}$, denoted by skel, also defines a discretization on X and yields the filtering $\downarrow q \mapsto F_{\downarrow q}X = \mathrm{skel}^{-1}\downarrow q$, $\downarrow q \in \mathsf{O}(\mathbb{N})$.

DEFINITION 2.29. A *CW-decomposition map* on X, denoted by $\mathrm{cell}\colon X \twoheadrightarrow \mathfrak{X}$, is a discretization map where each $\xi \in \mathfrak{X}$ is an q-cell for some q. Moreover, for every $\xi \in \mathfrak{X}$ there is a continuous map $f_\xi\colon \bar{B}^q \to X$, where $q = \dim \xi$, such that

(i) f_ξ restricts to a homeomorphism $f_\xi|_{B^q}\colon B^q \to |\xi|$;
(ii) $f_\xi(\bar{B}^q \smallsetminus B^q) \subset F_{\downarrow(q-1)}X$.

A subset $U \subset X$ is open (closed) if and only if $f_\xi^{-1}(U)$ is open (closed) in $\bar{B}^q$ for all $\xi \in \mathfrak{X}$. A CW-decomposition map is *regular* if the maps f_ξ are embeddings.

Note that if X admits a (finite) CW-decomposition map then X is a compact Hausdorff space, cf. [34]. Since X is Hausdorff, it follows that $\|\xi\| = f_\xi(\bar{B}^q)$.[29] CW-decompositions are general enough to include simplicial and cubical complexes. From a CW-decomposition map we can define the following finite topology in terms of a pre-order on $\mathfrak{X}$:

$$\xi \leqslant \xi' \quad \text{if and only if} \quad |\xi| \subset \mathrm{cl}|\xi'|,$$

which is called the *face partial order* on $\mathfrak{X}$.

LEMMA 2.30. *The pre-order $(\mathfrak{X}, \leqslant)$ is a partial order and the associated Alexandrov topology is a T_0 topology.*

PROOF. Suppose $\xi \neq \xi'$ and $\xi \sim \xi'$, i.e. $\xi \leqslant \xi'$ and $\xi' \leqslant \xi$, which implies that $\mathrm{cl}|\xi| = \mathrm{cl}|\xi'|$ and thus $f_\xi(\bar{B}^q) = f_{\xi'}(\bar{B}^q)$. Furthermore, $f_\xi(\bar{B}^q) = f_\xi(\bar{B}^q \smallsetminus B^q \cup B^q) =$

[29] By continuity $f_\xi(\bar{B}^q) \subset \mathrm{cl} f_\xi(B^q) = \|\xi\|$. On the other hand $\|\xi\| = \mathrm{cl} f_\xi(B^q) \subset \mathrm{cl} f_\xi(\bar{B}^q) = f_\xi(\bar{B}^q)$ since the continuous image of a compact set is compact and in a Hausdorff space compact sets are closed.

$f_\xi(\bar{B}^q \smallsetminus B^q) \cup |\xi|$, and similarly $f_{\xi'}(\bar{B}^q) = f_{\xi'}(\bar{B}^q \smallsetminus B^q) \cup |\xi'|$. By assumption $|\xi| \cap |\xi'| = \varnothing$, which yields that

$$|\xi| \subset f_{\xi'}(\bar{B}^q \smallsetminus B^q) \subset F_{\downarrow(q-1)}X, \quad |\xi'| \subset f_\xi(\bar{B}^q \smallsetminus B^q) \subset F_{\downarrow(q-1)}X.$$

This contradicts the definition of $F_{\downarrow(q-1)}X$ and the fact that all cells are realized as disjoint sets in X. Therefore $\xi \sim \xi'$ if and only if $\xi = \xi'$ and $\leqslant$ is a partial order and the Alexandrov topology is T_0. □

The next step is to show that $\mathrm{cell}\colon X \to (\mathfrak{X}, \leqslant)$ is a continuous discretization map, i.e., that $\leqslant$ is $\mathscr{T}$-consistent. Moreover, we also show that $\dim$ is order-preserving.

LEMMA 2.31. *Let* $\mathrm{cl}\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$ *be the closure operator defined by face partial order* $\leqslant$. *Then,*

(i) $\leqslant$ *is a* $\mathscr{T}$*-consistent partial order and* $\mathrm{cl}|\xi| = |\mathbf{cl}\,\xi|$ *for all* $\xi \in \mathfrak{X}$;
(ii) $\dim\colon (\mathfrak{X}, \leqslant) \to (\mathbb{N}, \leqslant)$ *is order-preserving.*

PROOF. We have that $\mathrm{cl}|\xi| = \|\xi\| = f_\xi(\bar{B}^q)$ and thus, as before using Definition 2.29(i)-(ii), $f_\xi(\bar{B}^q) = f_\xi(\bar{B}^q \smallsetminus B^q \cup B^q) = f_\xi(\bar{B}^q \smallsetminus B^q) \cup |\xi|$. This implies that $\mathrm{cl}|\xi|$ is a union of sets $|\xi'|$ and more precisely

$$\mathrm{cl}|\xi| = \bigcup\big\{|\xi'| \mid |\xi'| \subset \mathrm{cl}|\xi|\big\} = \bigcup\big\{|\xi'| \mid \xi' \leqslant \xi\big\} = \big|\mathrm{cl}\,\xi\big|,$$

which proves (i). As for (ii) we have that $\mathrm{cl}|\xi| \smallsetminus |\xi| = f_\xi(\bar{B}^q \smallsetminus B^q) = \bigcup\big\{|\xi'| \mid \xi' \leqslant \xi\big\} \subset F_{\downarrow(q-1)}X$ and thus $\dim \xi' \leqslant q-1$. Consequently, $\xi' \leqslant \xi$ implies $\dim \xi' \leqslant \dim \xi$. □

Lemma 2.31 shows in particular that cell is continuous open map and thus a discretization with respect to the face partial order. Moreover, $|\cdot|$ is an injective homomorphism of closure algebras. Indeed by additivity of $|\cdot|$ and cl we have that $\mathrm{cl}|\mathcal{U}| = |\mathbf{cl}\,\mathcal{U}|$ for all $\mathcal{U} \in \mathsf{Set}(\mathfrak{X})$. The associated CA-discretization $(\mathfrak{X}, \mathbf{cl}, |\cdot|)$ is Boolean and is called a *CW-decomposition* for X.

LEMMA 2.32. *A CW-decomposition map* $\mathrm{cell}\colon X \to \mathfrak{X}$ *is a natural discretization map.*[30]

PROOF. For every closed subset $\mathcal{U} \subset \mathfrak{X}$ the realization $|\mathcal{U}| \subset X$ is a sub CW-decomposition and therefore a deformation retract of a neighborhood in X, cf. [34, Prop. A.5.]. □

2.6.3. General closure and bi-closure algebras. A more general notion of closure algebra is given by a Boolean algebra $\mathsf{B} = (\mathsf{B}, \vee, \wedge, {}^{\neg})$ and an operator $\mathrm{cl}\colon \mathsf{B} \to \mathsf{B}$ (an abstract closure operator) satisfying Axiom (K1)-(K4).[31] The Boolean algebra B is not necessarily complete, nor atomic. Such algebras are referred to as

[30] cf. Defn. 2.17.

[31] Replace $\varnothing$ and X by the neutral elements 0 and 1 respectively, as well as the binary operations and complement.

closure algebras and are denoted as $(\mathsf{B}, \mathrm{cl})$. In [51] various representation results for general closure algebras are given. For a closure algebra the lattice of closed elements is given by $\mathsf{Fwdset}(\mathrm{cl}) = \{b \in \mathsf{B} \mid \mathrm{cl}\ b \leqslant b\}$. The latter can be considered as *generalized topological space*, q.v. [55]. In terms of closed sets this entails that $\mathsf{Fwdset}(\mathrm{cl})$ is a bounded lattice, cf. Prop. 2.1 and if $\bigwedge b$, $b \in \mathsf{Fwdset}(\mathrm{cl})$, exists in B, then $\bigwedge b \in \mathsf{Fwdset}(\mathrm{cl})$. Moreover, the expression $\max\{a \mid b \leqslant a,\ \mathrm{cl}\ a = a\}$ exists in B. By definition the latter satisfies $b \leqslant \max\{a \mid b \leqslant a,\ \mathrm{cl}\ a = a\} \leqslant \mathrm{cl}\ b$ and thus it exists and $\mathrm{cl}\ b = \bigwedge\{a \mid b \leqslant a,\ \mathrm{cl}\ a = a\}$. The generalized topological space is denoted $\big(\mathsf{B}, \mathsf{Fwdset}(\mathrm{cl})\big)$. Closure algebras are equivalent to generalized topological spaces. In a similar fashion a *bi-closure algebra* is given by a Boolean algebra and two abstract closure operators $\mathrm{cl}, \mathrm{cl}'\colon \mathsf{B} \to \mathsf{B}$ and is denoted by $(\mathsf{B}, \mathrm{cl}, \mathrm{cl}')$. For bi-closure algebras we have an associated generalized bi-topological space. The same consideration hold if we used derivative operators.

For bi-closure algebras discretization can be formulated as bofore. An embedding

$$|\cdot|\colon \big(\mathsf{Set}(\mathfrak{X}), \mathbf{cl}, \mathbf{cl}'\big) \rightarrowtail \big(\mathsf{B}, \mathrm{cl}, \mathrm{cl}'\big),$$

is a bi-topological CA-discretization if $\mathrm{cl}|\mathcal{U}| \subset |\mathbf{cl}\mathcal{U}|$ and $\mathrm{cl}'|\mathcal{U}| \subset |\mathbf{cl}'\mathcal{U}|$ for all $\mathcal{U} \subset \mathfrak{X}$. In order to describe discretization in terms of continuous discretization maps we need to use a representation of the bi-closure algebra such as the approach by Mckinsey-Tarski, cf. [51], or Jonsson-Tarski, cf. [38].

In a slightly more general setting one may define a *modal algebra* by specifying a Boolean algebra B and an operator $\Phi\colon \mathsf{B} \to \mathsf{B}$ satisfying (M1)-(M2).[32] The latter is called a(n) (abstract) modal operator and the associated modal algebra is denoted by (B, Φ). For a modal algebra the lattice of closed elements is given by $\mathsf{Fwdset}(\Phi) = \{b \in \mathsf{B} \mid \Phi b \leqslant b\}$. Similarly a *bi-modal algebra* is given by two modal operators $\Phi, \Phi'\colon \mathsf{B} \to \mathsf{B}$ and is denoted by $(\mathsf{B}, \Phi, \Phi')$. Even though the closed element almost yield generalized topological spaces the correspondence is more involved in this case.

[32] As for closure algebras use the neutral elements 0 and 1.

CHAPTER 3

Flow topologies and discretization of dynamics

In the previous sections we discussed topological and bi-topological spaces in terms of closure algebras which is the appropriate language for formalizing discretization of topology. The next step is to model the dynamics of semi-flows on topological spaces via appropriately constructed topologies on X. Such topologies may be realized in many different ways and we refer to the dynamics induced topologies as *flow topologies*. The objective is not to develop explicit methods for discretizing dynamics but to describe the contours of a theory that discretizes dynamics in terms of discretizing two topologies.

3.1. Dynamics as topology

As discussed in the previous sections a topological space $(X, \mathscr{T})$ can be equivalently described via the closure algebra $\mathrm{cl}\colon \mathsf{Set}(X) \to \mathsf{Set}(X)$. This description is convenient for introducing new topologies in relation to dynamical systems.

3.1.1. Basic flow topologies. For a semi-flow φ define the (completely additive) modal operators $\Gamma^-, \Gamma^+\colon \mathsf{Set}(X) \to \mathsf{Set}(X)$ given by

$$U \mapsto \Gamma^- U := \bigcup_{t>0} \varphi(-t, U), \quad U \mapsto \Gamma^+ U := \bigcup_{t>0} \varphi(t, U), \tag{3.1}$$

The operators Γ^- and Γ^+ which are called the *strict backward image* and *strict forward image* operators respectively. The operators $\mathrm{cl}^- = \mathrm{id} \cup \Gamma^-$ and $\mathrm{cl}^+ = \mathrm{id} \cup \Gamma^+$ are obtained by taking $t \geqslant 0$ and satisfy all four Kuratowski axioms (K1)-(K4) for closure operators. The derivative operators Γ^- and Γ^+ define the topologies $\mathscr{T}^-$ and $\mathscr{T}^+$ on X respectively, which are Alexandrov topologies on X. The associated specialization pre-order on X defined by $\mathscr{T}^+$ will be denoted by $\leqslant^+$ and is defined by $y \leqslant^+ x$ if and only if $y \in \mathrm{cl}^+\{x\}$. The latter is characterized by

$$y \leqslant^+ x \quad \text{if and only if} \quad y = \varphi(t, x) \quad \text{for some} \quad t \geqslant 0.$$

The pre-order $\leqslant^+$ does record the directionality of the flow ϕ but discards the sense of time and invariance. The closed sets in $\mathscr{T}^+$ are the forward invariant sets for ϕ and are denoted by $\mathsf{Invset}^+(\varphi)$. As a matter of fact, using the notation in Sect. 2.2, we have that $\mathsf{Invset}^+(\varphi) = \mathsf{Fwdset}(\Gamma^+) = \mathsf{Fwdset}(\mathrm{cl}^+)$. The two topologies $\mathscr{T}$ and $\mathscr{T}^+$ combined comprise the bi-topological space $(X, \mathscr{T}, \mathscr{T}^+)$. The associated specialization pre-order on X defined by $\mathscr{T}^-$ will be denoted by $\leqslant^-$ and is defined by $y \leqslant^- x$ if and only if $y \in \mathrm{cl}^-\{x\}$. This pre-order is characterized by $y \leqslant^- x$ if and only if $y \in \varphi(-t,x)$ for some $t \geqslant 0$, i.e. $x = \varphi(t,y)$ for some $t \geqslant 0$. This shows that $\leqslant^-$ is the opposite pre-order to $\leqslant^+$ and closed sets in $\mathscr{T}^+$ are open sets in $\mathscr{T}^-$ and vice versa. The Alexandrov topologies $\mathscr{T}^-$ and $\mathscr{T}^+$ are each other's opposites. The closed sets in $\mathscr{T}^-$ are the backward invariant sets for φ and are denoted by $\mathsf{Invset}^-(\varphi)$. The two topologies $\mathscr{T}$ and $\mathscr{T}^-$ combined comprise the bi-topological space $(X, \mathscr{T}, \mathscr{T}^-)$.

REMARK 3.1. For the topologies $\mathscr{T}^-$ and $\mathscr{T}^+$ the closure and conjugate closure operators are related: $\bar{\mathrm{cl}}^+ = \mathrm{star}^+ = \mathrm{cl}^-$ and $\bar{\mathrm{cl}}^- = \mathrm{star}^- = \mathrm{cl}^+$.

Since cl^- and cl^+ are closure operators Γ^- and Γ^+ are (canonical) *derivative operators* satisfying the Axioms (D1)-(D3). Observe that

$$\begin{aligned}\Gamma^+\big(\Gamma^+U\big) &= \bigcup_{t>0}\varphi\Big(t, \bigcup_{s>0}\varphi(s,U)\Big) = \bigcup_{t>0}\bigcup_{s>0}\varphi(t,\varphi(s,U)) \\ &= \bigcup_{s+t>0}\varphi(s+t,U) = \Gamma^+U,\end{aligned} \tag{3.2}$$

and therefore Γ^+ satisfies the stronger idempotency axiom (K3), i.e. $\Gamma^+(\Gamma^+U) = \Gamma^+U$ and Γ^+ is an idempotent derivative operator. In the same way one proves that Γ^- satisfies the idempotency axiom in (K3).

The derivative operator Γ^+ is associated with a binary relation on X: $y <^+ x$ if and only if $y = \varphi(t,x)$ for some $t > 0$. Observe that the $<^+$ is a transitive relation and the reflexive closure yields the specialization pre-order $\leqslant^+$. Points $x \in X$ for which $\Gamma^+\{x\} = \{x\}$ correspond to fixed points of ϕ and are examples of reflexive points for $<^+$, i.e. $x <^+ x$. Other reflexive points are given by periodic orbits for ϕ, cf. [1]. The derivative operator Γ^+ does not detect invariant sets in general. Indeed, for $\varphi(t,x) = x+t$ the set $U = (0,\infty)$ satisfies $\Gamma^+U = U$ but is not invariant. To capture invariance one can use τ-forward image operator Γ^+_τ and topology $\mathscr{T}^+_\tau$ by considering forward images from $t \geqslant \tau$ and similarly for τ negative. cf. Sect. 6.1. In view of the considerations in Sect. 2.2 one can also consider the topology $\mathscr{T}_\tau$ defined via the modal operator Φ defined by $\Phi U = \varphi(\tau, U)$ for some $\tau \neq 0$. In particular we have that

$$\mathscr{T}^+ \subset \mathscr{T}^+_\tau \subset \mathscr{T}_\tau, \quad \tau > 0,$$

and the same for $\tau < 0$ and all topologies are Alexandrov. The flow topologies capture directionality but do not require any continuity properties on ϕ. An interesting feature of topological dynamics is to study convergence and decompositions. To do so we will now explore an alternative way to recast dynamics in terms of topology.

REMARK 3.2. If φ is a continuous semi-flow then the continuity of $\varphi(t,\cdot)$ with respect to the topology $\mathscr{T}^+$ is immediate. Continuity of φ with respect to $\mathscr{T}$ yields interaction of the two topologies as a manifestation of the continuous semi-flow φ on X. This implies the *subcommutativity* for $\mathscr{T}$ and $\mathscr{T}^+$, i.e. $U \subset X$ $\mathscr{T}^+$-closed implies that $\mathrm{cl}\, U$ is $\mathscr{T}^+$-closed. This makes the space $(X, \mathscr{T}, \mathscr{T}^+)$ a $(\mathscr{T}, \mathscr{T}^+)$-subcommutative bi-topological space.

3.1.2. The block-flow topology $\mathscr{T}_{\blacksquare}^{-}$. As our aim is an algebraization of dynamics that recovers invariance based on Wazewski's principle, cf. [12], we study the attracting and repelling blocks of φ. Recall that closed *attracting blocks* are defined by

$$\mathsf{ABlock}_{\mathscr{C}}(\varphi) := \big\{U \subset X \mid \mathrm{cl}\, U = U,\ \varphi(t, U) \subset \mathrm{int}\, U,\ \forall t > 0\big\}. \tag{3.3}$$

Let $U, U' \in \mathsf{ABlock}_{\mathscr{C}}(\varphi)$. If the singular homology satisfies $H(U, U') \neq 0$, then $\mathrm{Inv}(U \smallsetminus U') \neq \varnothing$, cf. Sect. 4.5. In order to incorporate attracting blocks into the theory of (bi)-closure algebras we construct topologies derived from the basic flow topologies. In general, there are a number of options to define such topologies. We highlight one particular choice that serves the purpose of constructing closed attracting blocks, cf. Sect. 6.1.

LEMMA 3.3. *The operator* $\Gamma_{\blacksquare}^{-} := \Gamma^{-}\mathrm{cl}$ *is a modal operator.*

PROOF. Axioms (M1)-(M2) in Section 2.2 are satisfied since cl is a closure operator and Γ^{-} is a derivative operator, which establishes $\Gamma_{\blacksquare}^{-}$ as a normal, additive operator on $\mathsf{Set}(X)$. □

The operator $\Gamma_{\blacksquare}^{-}$ does not define a derivative operator since (D3) is not satisfied in general. Using the theory in Section 2.2 $\Gamma_{\blacksquare}^{-}$ can be used to define a topology on X.

DEFINITION 3.4. A subset $U \subset X$ is closed with respect to $\Gamma_{\blacksquare}^{-}$ if and only if $\Gamma_{\blacksquare}^{-}U \subset U$. Such set are denoted by $\mathsf{Fwdset}(\Gamma_{\blacksquare}^{-})$.

By Proposition 2.1 $\mathsf{Fwdset}(\Gamma_{\blacksquare}^{-})$ defines a topology on X which is denoted by $\mathscr{T}_{\blacksquare}^{-}$ and is called the *block-flow topology* on X. Observe that the condition that $U \subset X$ is $\mathscr{T}_{\blacksquare}^{-}$-closed is equivalent to the condition $\varphi(-t, \mathrm{cl}\, U) \subset U$ for all $t > 0$. The associated closure operator is given by

$$\mathrm{cl}_{\blacksquare}^{-} U := \bigcap \big\{U' \supset U \mid U' \in \mathsf{Fwdset}(\Gamma_{\blacksquare}^{-})\big\}. \tag{3.4}$$

By definition the block-flow topology $\mathscr{T}_{\blacksquare}^{-}$ is not necessarily an Alexandrov topology. A subset $U \subset X$ is $\mathscr{T}_{\blacksquare}^{-}$*-closed* if $\mathrm{cl}_{\blacksquare}^{-} U = U$.

PROPOSITION 3.5. *The block-flow topology* $\mathscr{T}_{\blacksquare}^{-}$ *is a coarsening of the Alexandrov topology* $\mathscr{T}^{-}$ *which is generated by* cl^{-}*, i.e.* $\mathrm{cl}^{-} \subset \mathrm{cl}_{\blacksquare}^{-}$.

PROOF. By definition $\Gamma^{-}U \subset \Gamma_{\blacksquare}^{-}U \subset U$ which implies $\Gamma^{-}U \subset U$. Therefore, $\mathrm{cl}^{-} \subset \mathrm{cl}_{\blacksquare}^{-}$. □

PROPOSITION 3.6. *The maps* $\varphi(t, \cdot)$ *are continuous in the block-flow topology* $\mathscr{T}_{\blacksquare}^{-}$ *for all* $t \geqslant 0$.

PROOF. For the backward image it holds that $\varphi(-t, \Gamma^{-}U) = \Gamma^{-}\varphi(-t, U)$, $t \geqslant 0$, and thus $\varphi(-t, \Gamma^{-}\mathrm{cl}\, U) = \Gamma^{-}\varphi(-t, \mathrm{cl}\, U) \supset \Gamma^{-}\mathrm{cl}\, \varphi(-t, U)$. Suppose U is $\mathscr{T}_{\blacksquare}^{-}$-closed, i.e. $\Gamma_{\blacksquare}^{-}U \subset U$. Then,

$$\Gamma_{\blacksquare}^{-}\varphi(-t, U) \subset \varphi(-t, \Gamma_{\blacksquare}^{-}U) \subset \varphi(-t, U),$$

which proves that $\varphi(-t,U)$ is $\mathscr{T}_{\blacksquare}^{-}$-closed. The latter proves that for every $t \geqslant 0$ the map $\varphi(t,\cdot)$ is continuous in the $\mathscr{T}_{\blacksquare}^{-}$-flow topology. $\square$

Observe, by Proposition 3.6, that $\varphi(t,\cdot)$ is continuous in both topologies $\mathscr{T}$ and $\mathscr{T}_{\blacksquare}^{-}$. The block-flow topology $\mathscr{T}_{\blacksquare}^{-}$ defines a new, derived subcommutative bi-topological space $(X,\mathscr{T},\mathscr{T}_{\blacksquare}^{-})$. Open and closed sets in the block-flow topology $\mathscr{T}_{\blacksquare}^{-}$ can be characterized via the semi-flow φ.

LEMMA 3.7. *A subset $U \subset X$ is closed in the block-flow topology $\mathscr{T}_{\blacksquare}^{-}$ if and only if*

$$\varphi(-t,\operatorname{cl} U) \subset U, \quad \forall t > 0. \tag{3.5}$$

Similarly, a subset $U \subset X$ is open in the block-flow topology $\mathscr{T}_{\blacksquare}^{-}$ if and only if

$$\varphi(t,U) \subset \operatorname{int} U, \quad \forall t > 0. \tag{3.6}$$

PROOF. By definition a subset $U \subset X$ is $\mathscr{T}_{\blacksquare}^{-}$-closed if and only if $\operatorname{cl}_{\blacksquare}^{-} U = U$ which implies $\Gamma_{\blacksquare}^{-} U \subset U$ and thus $\Gamma^{-}\operatorname{cl} U \subset U$. The latter implies that $\varphi(-t,\operatorname{cl} U) \subset U$ for all $t > 0$. Conversely, if (3.5) holds then $\Gamma^{-}\operatorname{cl} U \subset U$ and thus $\operatorname{cl}_{\blacksquare}^{-} U = U$.

By definition a subset $U \subset X$ is $\mathscr{T}_{\blacksquare}^{-}$-open if and only if U^c is $\mathscr{T}_{\blacksquare}^{-}$-closed, i.e. $\varphi(-t,\operatorname{cl} U^c) \subset U^c$ for all $t > 0$. The latter is equivalent to $\varphi(-t,\operatorname{cl} U^c)^c \supset U$ for all $t > 0$, which results in the equivalent statement that $\varphi(-t,\operatorname{int} U) \supset U$ for all $t > 0$. If we compose the latter with $\varphi(t,\cdot)$ we obtain

$$\varphi(t,U) \subset \varphi\big(t,\varphi(-t,\operatorname{int} U)\big) \subset \operatorname{int} U, \quad \forall t > 0.$$

On the hand, composition of (3.6) with the inverse image $\varphi(-t,\cdot)$ gives

$$U \subset \varphi\big(-t,\varphi(t,U)\big) \subset \varphi(-t,\operatorname{int} U), \quad \forall t > 0,$$

which prove that U is $\mathscr{T}_{\blacksquare}^{-}$-open. $\square$

The $(\mathscr{T},\mathscr{T}_{\blacksquare}^{-})$-pairwise clopen sets in $(X,\mathscr{T},\mathscr{T}_{\blacksquare}^{-})$ are defined as sets $U \subset X$ that are closed in $\mathscr{T}$ and open in $\mathscr{T}_{\blacksquare}^{-}$.

THEOREM 3.8. *A subset $U \subset X$ is a closed attracting block, cf.* (3.3)*, if and only if U is a $(\mathscr{T},\mathscr{T}_{\blacksquare}^{-})$-pairwise clopen set in $(X,\mathscr{T},\mathscr{T}_{\blacksquare}^{-})$.*

PROOF. This follows from Lemma 3.7 and the definition of closed attracting blocks, cf. Eqn. (3.3). $\square$

The set of closed attracting blocks $\mathsf{ABlock}_{\mathscr{C}}(\varphi)$ is a sublattice of $\mathsf{Set}(X)$, cf. [42, 43], [40]. This lattice is not complete in general. The $(\mathscr{T}_{\blacksquare}^{-},\mathscr{T})$-pairwise clopen sets correspond to open repelling blocks. Indeed, $|\mathcal{U}|$ is open and (3.6) we have that $\varphi(-t,\operatorname{cl}|\mathcal{U}|) \subset |\mathcal{U}|$ for all $t > 0$. We denote the open *repelling blocks* by $\mathsf{RBlock}_{\mathscr{O}}(\varphi)$.

REMARK 3.9. The continuity of φ in the $\mathscr{T}$-topology can be relaxed to an $\mathbb{R}^+$-parameter family $\varphi(t,\cdot)$ of continuous maps in the $\mathscr{T}$-topology for all $t \geqslant 0$. This implies in particular that we can apply discretization and topologization to other families of maps such as $t \in \mathbb{Z}^+$ which is equivalent to iterating a map, i.e. discrete time dynamics. Continuity of $\varphi(t,\cdot)$ implies a more equal role for both topologies.

The continuity of φ on $\mathbb{R}^+ \times X$ comes in in two instances, (i) equivalent discretization via condensed Morse pre-orders , cf. Sect. 3.3, and (ii) algebraization in order to invoke Wazewski's principle for finding invariant sets, cf. Sect. 6.1 and [12].

REMARK 3.10. If we consider variations of the operator $\Gamma^-_\blacksquare$ such as $\Gamma^+_\blacksquare = \Gamma^+ \mathrm{cl}$ then the $(\mathscr{T}_\blacksquare^-, \mathscr{T})$-clopen sets are open attracting blocks and the $(\mathscr{T}, \mathscr{T}_\blacksquare^-)$-clopen sets are the closed repelling blocks. Other variations entail $\mathrm{cl}\ \Gamma^+$ and $\mathrm{cl}\ \Gamma^-$, cf. Rem. 3.2 and Sect. 6.1. Our definition of block-flow topology is suitable for the theory in this text.

3.2. Discretization of the block-flow topology

In Section 2.3 we discussed discretization of topology in terms of closure algebras and derivative algebras. The standard CW-decompositions of spaces are examples of such discretizations. In this section we apply the closure algebra discretization to the block-flow topology which provides the appropriate discretization of dynamics. As we have encoded φ as a topological space $(X, \mathscr{T}, \mathscr{T}_\blacksquare^-)$, we can apply the tools from Section 2.3, e.g., CA-discretizations, MA-discretizations, discretization maps and topology consistent pre-orders.

Let $\mathfrak{X}$ be a finite sets and let $\mathbf{cl}^-_\blacksquare : \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$ be a closure operator such that $\mathrm{cl}^-_\blacksquare|\mathcal{U}| \subset |\mathbf{cl}^-_\blacksquare \mathcal{U}|$ for all $\mathcal{U} \in \mathsf{Set}(\mathfrak{X})$. This induces a pre-order $\leqslant^-_\blacksquare$ by (2.8) and yields the continuity of $\mathrm{disc}\colon (X, \mathscr{T}_\blacksquare^-) \to (\mathfrak{X}, \leqslant^-_\blacksquare)$ defined in Eqn. (2.12). Conversely, if $\leqslant^-_\blacksquare$ is any $\mathscr{T}_\blacksquare^-$-consistent pre-order with respect to disc then the associated closure operator $\mathbf{cl}^-_\blacksquare$ defines a CA-discretization for $(X, \mathscr{T}_\blacksquare^-)$.

LEMMA 3.11. *A pre-order $\leqslant^-_\blacksquare$ on $\mathfrak{X}$ with associated closure operator $\mathbf{cl}^-_\blacksquare : \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$ is $\mathscr{T}_\blacksquare^-$-consistent with respect to $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ if and only if*

$$\varphi\big(-t, \mathrm{cl}|\xi|\big) \subset \big|\mathbf{cl}^-_\blacksquare \xi\big|, \quad \forall t > 0, \quad \textit{and} \quad \forall \xi \in \mathfrak{X}. \tag{3.7}$$

PROOF. The discretization $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ is continuous if and only if $\mathcal{U}$ closed in $(\mathfrak{X}, \leqslant^-_\blacksquare)$ implies that $|\mathcal{U}|$ is $\mathscr{T}_\blacksquare^-$-closed. Assume (3.7) is satisfied. Let $\mathbf{cl}^-_\blacksquare \mathcal{U} = \mathcal{U}$, then

$$\begin{aligned}\varphi(-t, \mathrm{cl}|\mathcal{U}|) &= \bigcup_{\xi \in \mathcal{U}} \varphi(-t, \mathrm{cl}|\xi|) \subset \bigcup_{\xi \in \mathcal{U}} |\mathbf{cl}^-_\blacksquare \xi| \\ &= \Big|\mathbf{cl}^-_\blacksquare\Big(\bigcup_{\xi \in \mathcal{U}} \{\xi\}\Big)\Big| = |\mathbf{cl}^-_\blacksquare \mathcal{U}| = |\mathcal{U}|, \quad \forall t > 0,\end{aligned}$$

which proves that $\Gamma^-_\blacksquare|\mathcal{U}| \subset |\mathcal{U}|$ and thus $|\mathcal{U}|$ is $\mathscr{T}_\blacksquare^-$-closed. Conversely, if disc is continuous, then $\mathcal{U}$ closed in $(\mathfrak{X}, \leqslant^-_\blacksquare)$ implies $|\mathcal{U}|$ is $\mathscr{T}_\blacksquare^-$-closed and therefore $\varphi(-t, \mathrm{cl}|\mathcal{U}|) \subset |\mathcal{U}|$. Choose $\mathcal{U} = \mathbf{cl}^-_\blacksquare \xi$. This implies that

$$\varphi(-t, \mathrm{cl}|\xi|) \subset \varphi(-t, \mathrm{cl}|\mathbf{cl}^-_\blacksquare \xi|) \subset |\mathbf{cl}^-_\blacksquare \xi|, \quad \forall t > 0, \quad \text{and} \quad \forall \xi \in \mathfrak{X}.$$

which establishes (3.7). □

In the discrete setting we can define the discrete analogues of the block-flow topology $(X, \mathscr{T}_\blacksquare^-)$ via discretizations of $\mathscr{T}$ and $\mathscr{T}^-$. Let $\mathbf{\Gamma}^-\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$ be a discrete derivative for $\mathscr{T}^-$ and define the additive operator $\mathbf{\Gamma}^-_{\blacksquare} := \mathbf{\Gamma}^-\mathbf{cl}\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$.

LEMMA 3.12. *The triple* $(\mathsf{Set}(\mathfrak{X}, \mathbf{\Gamma}^-_{\blacksquare}, |\cdot|)$ *defines a MA-discretization of* $(X, \mathscr{T}^-_{\blacksquare})$. *The associated discrete closure operator is given by* $\mathbf{cl}^-_{\blacksquare} = \bigcup_{k\geqslant 0}\big(\mathbf{\Gamma}^-_{\blacksquare}\big)^k$ *and the triple* $(\mathfrak{X}, \mathbf{cl}^-_{\blacksquare}, |\cdot|)$ *is a CA-discretization for* $(X, \mathscr{T}^-_{\blacksquare})$.

PROOF. In order to establish $(\mathsf{Set}(\mathfrak{X}, \mathbf{\Gamma}^-_{\blacksquare}, |\cdot|)$ as a MA-discretization we use the fact that $\mathbf{cl}$ defines a CA-discretization $(\mathfrak{X}, \mathbf{cl}, |\cdot|)$ for $(X, \mathscr{T})$ and $\mathbf{\Gamma}^-$ defines a MA-discretization $(\mathfrak{X}, \mathbf{\Gamma}^-, |\cdot|)$ for $(X, \mathscr{T}^-)$. This implies, for $\mathcal{U} \subset \mathfrak{X}$, that

$$\Gamma^-_{\blacksquare}|\mathcal{U}| = \Gamma^-\mathrm{cl}|\mathcal{U}| \subset \Gamma^-|\mathbf{cl}\mathcal{U}| \subset |\mathbf{\Gamma}^-\mathbf{cl}\,\mathcal{U}| = |\mathbf{\Gamma}^-_{\blacksquare}\mathcal{U}|, \tag{3.8}$$

which by Proposition 2.11 shows that $(\mathsf{Set}(\mathfrak{X}, \mathbf{\Gamma}^-_{\blacksquare}, |\cdot|)$ is a MA-discretization of $(X, \mathscr{T}^-_{\blacksquare})$ and provides the expression for $\mathbf{cl}^-_{\blacksquare}$. It remains to show that $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ is continuous. Let $\mathcal{U} = \{\xi\}$, then (3.8) yields $\varphi(-t, \mathrm{cl}|\xi|) \subset \Gamma^-_{\blacksquare}|\xi| \subset |\mathbf{\Gamma}^-_{\blacksquare}\xi| \subset |\mathbf{cl}^-_{\blacksquare}\xi|$ for all $t > 0$ which, by Lemma 3.11, proves that disc continuous. □

The following lemma formulates a criterion for discretizing the block-flow topology $\mathscr{T}^-_{\blacksquare}$ without using a discretization for $\mathscr{T}^-$.

LEMMA 3.13. *Let* $\mathbf{\Phi}\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$ *be a modal operator such that*

$$\varphi\big(-t, \mathrm{cl}|\xi|\big) \subset \big|\mathbf{\Phi}\xi\big|, \quad \forall t > 0, \quad \textit{and} \quad \forall \xi \in \mathfrak{X}. \tag{3.9}$$

Then, the operator $\mathbf{cl}^-_{\blacksquare} = \mathbf{\Phi}^{+=} = \bigcup_{k\geqslant 0}\mathbf{\Phi}^k\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$ *is a closure operator and yields a CA-discretization* $(\mathfrak{X}, \mathbf{cl}^-_{\blacksquare}, |\cdot|)$ *of the block-flow topology* $(X, \mathscr{T}^-_{\blacksquare})$.

PROOF. The fact that $\mathbf{\Phi}^{+=}$ is closure operator follows from Lemma 3.12. By assumption $\bigcup_{t>0}\varphi(-t, \mathrm{cl}|\xi|) = \Gamma^-_{\blacksquare}|\xi| \subset |\mathbf{\Phi}\xi|$. As in the proof of Lemma 3.12 this implies that $\varphi(-t, \mathrm{cl}|\xi|) \subset \Gamma^-_{\blacksquare}|\xi| \subset |\mathbf{\Phi}\xi| \subset |\mathbf{\Phi}^{+=}\xi| \subset |\mathbf{cl}^-_{\blacksquare}\xi|$ for all $t > 0$ which proves that disc is continuous, completing the proof. □

The following result gives a local version of the above criterion and provides a practical method for constructing CA-discretizations for the block-flow topology. Fig. 3.1 displays an example of a discretization of both $(X, \mathscr{T})$ and $(X, \mathscr{T}^-_{\blacksquare})$.

THEOREM 3.14. *Let* $(X, \mathscr{T})$ *be compact and let* $\mathbf{\Phi}\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$ *be a modal operator. Assume that for every* $\xi \in \mathfrak{X}$ *there exists* $t_\xi > 0$ *such that*

$$\varphi\big(-t, \mathrm{cl}|\xi|\big) \subset \big|\mathbf{\Phi}\xi\big|, \quad \forall 0 < t \leqslant t_\xi, \quad \textit{and} \quad \forall \xi \in \mathfrak{X}. \tag{3.10}$$

Then, $(\mathfrak{X}, \mathbf{cl}^-_{\blacksquare}, |\cdot|)$*, with* $\mathbf{cl}^-_{\blacksquare} = \mathbf{\Phi}^{+=}$*, is a CA-discretization for* $(X, \mathscr{T}^-_{\blacksquare})$.

PROOF. The proof of based on the following observation. By the compactness of X we may assume, without loss of generality, that $t_\xi \geqslant t_* > 0$ for all $\xi \in \mathfrak{X}$. Then, $\varphi(-t, \mathrm{cl}|\xi|) \subset |\mathbf{\Phi}\xi|$ for all $0 < t \leqslant t_*$ and for all $\xi \in \mathfrak{X}$. Observe that

$$\begin{aligned}\varphi(-2t, \mathrm{cl}|\xi|) = \varphi\big(-t, \varphi(-t, \mathrm{cl}|\xi|)\big) &\subset \varphi\big(-t, |\mathbf{\Phi}\xi|\big)\\ &\subset \varphi\big(-t, \mathrm{cl}|\mathbf{\Phi}\xi|\big) \subset |\mathbf{\Phi}^2\xi|,\end{aligned}$$

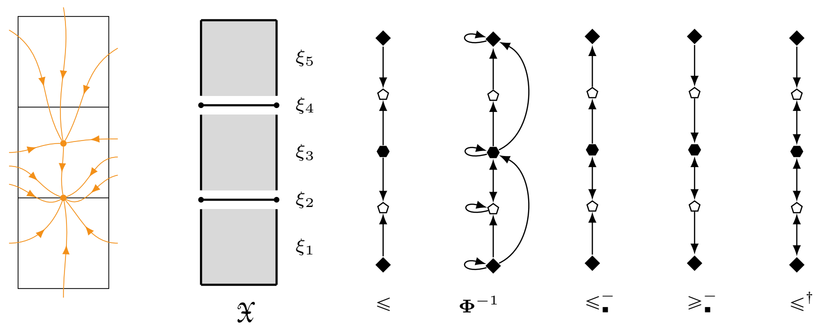

FIGURE 3.1. The topological space $(X, \mathscr{T})$ and semi-flow ϕ [left 1]. A discretization of X with $\mathscr{T}$-consistent pre-order $(\mathfrak{X}, \leqslant)$ [middle 2, 3]. The relation $\mathbf{\Phi}^{-1}$ [middle 4] which generates the $\mathscr{T}_{\blacksquare}^{-}$-consistent pre-order $(\mathfrak{X}, \leqslant^-_{\blacksquare})$ [right 5]. The discretization map $\mathrm{disc}\colon X \to \mathfrak{X}$ is continuous with respect to both topologies. Common coarsening of $(\mathfrak{X}, \leqslant)$ and $(\mathfrak{X}, \geqslant^-_{\blacksquare})$ [right 6] resulting in a Morse pre-order $(\mathfrak{X}, \leqslant^{\dagger})$ [right 7]. All pre-orders are represented by their Hasse diagrams.

which yields $\varphi(-kt, \mathrm{cl}|\xi|) \subset |\mathbf{\Phi}^k\xi| \subset |\mathbf{\Phi}^{+=}\mathcal{U}|$, for all $k \geqslant 0$ and for all $0 < t \leqslant t_*$. As in the proof of Lemma 3.13, $\varphi(-t, \mathrm{cl}|\xi|) \subset |\mathbf{\Phi}^{+=}\xi| \subset |\mathbf{cl}^-_{\blacksquare}\xi|$ for all $t > 0$ which proves by Lemma 3.11 that disc is continuous, and thus $(\mathfrak{X}, \mathbf{\Phi}^{+=}, |\cdot|)$ is a CA-discretization for $(X, \mathscr{T}_{\blacksquare}^{-})$. $\square$

REMARK 3.15. The operator $\mathbf{\Phi}$ defines a relation on $\mathfrak{X}$: $(\eta, \xi) \in \phi$ if and only if $\eta \in \mathbf{\Phi}\xi$, cf. App. A.3. The transitive, reflexive closure of ϕ is the pre-order $\leqslant^-_{\blacksquare}$ associated to $\mathbf{cl}^-_{\blacksquare} := \mathbf{\Phi}^{+=}$.

REMARK 3.16. For discretizing the flow topologies $\mathscr{T}^-$ and $\mathscr{T}^+$ we can use the criteria in Lemmas 3.11 and 3.9 by discarding the topology $\mathscr{T}$, i.e. take cl to be the identity map. For example a discrete closure operator $\mathbf{cl}^+\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$ yields a CA-discretization for $\mathscr{T}^+$ if and only if $\varphi(t, |\xi|) \subset |\mathbf{cl}^+\xi|$ for all $t \geqslant 0$ and for all $\xi \in \mathfrak{X}$, i.e. $\mathscr{T}^+$-consistency for the associated pre-order $\leqslant^+$.

REMARK 3.17. The definition of the block-flow topology in Section 3.1.2 uses the modal operator $\Gamma^-_{\blacksquare}$. This construction works for any modal operator Φ on $\mathsf{Set}(X)$ as is explained in Section 2.2.

3.3. Morse pre-orders

In this section we explain the implications of discretization with respect to two topologies in the sense of $(\mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$- pairwise clopen sets. We consider the bi-topological space $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$ and we use the theory in Section 2.5 to discuss discretization in this setting. Let $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ be a discretization map and let $(\mathfrak{X}, \leqslant^{\dagger})$ be an anatagonistic pre-order for $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$, which motivates the following definition:

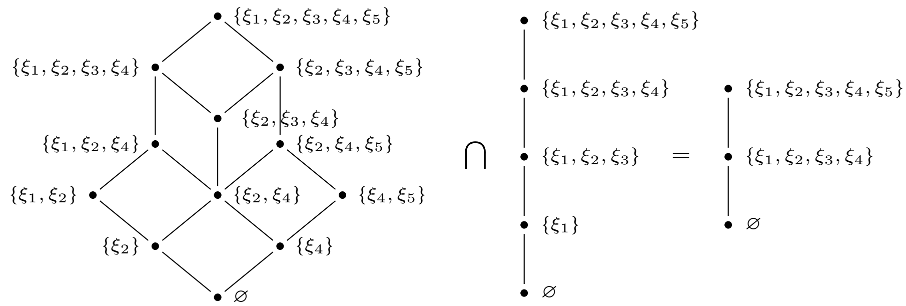

FIGURE 3.2. Intersection of both the lattice of down-sets $\mathsf{O}(\mathfrak{X}, \leqslant)$ and up-sets $\mathsf{U}(\mathfrak{X}, \leqslant_{\blacksquare}^{-})$ for Fig. 3.1 yields the coarsening $(\mathsf{SC}, \leqslant)$ [right] of the Morse pre-order $(\mathfrak{X}, \leqslant^{\dagger})$.

DEFINITION 3.18. Let $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ be a discretization map. A *Morse pre-order* on $\mathfrak{X}$ is an anatagonistic pre-order $\leqslant^{\dagger}$ for $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$, i.e.

(i) $\leqslant^{\dagger}$ is $\mathscr{T}$-consistent with respect to disc;
(ii) $\leqslant^{\dagger}$ is $\mathscr{T}_{\blacksquare}^{-}$-co-consistent with respect to disc.

The associated closure operator is denoted by $\mathrm{cl}^{\dagger}\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$.

By Theorem 2.22 $(\mathfrak{X}, \leqslant^{\dagger})$ is an antagonistic coarsening of discretizations for both $\mathscr{T}$ and $\mathscr{T}_{\blacksquare}^{-}$, cf. Sect. 2.5. The conditions for an antagonistic pre-order imply that for $\mathcal{U} \subset \mathfrak{X}$ we have that (i) $\mathrm{cl}|\mathcal{U}| \subset |\mathbf{cl}^{\dagger}\mathcal{U}|$ and (ii) $\mathrm{cl}_{\blacksquare}^{-}|\mathcal{U}| \subset |\,\mathbf{star}^{\dagger}\,\mathcal{U}|$. In particular, if $\mathcal{U} \in \mathsf{O}(\mathfrak{X}, \leqslant^{\dagger})$, then $\mathbf{cl}^{\dagger}\mathcal{U} = \mathcal{U}$ and thus $|\mathcal{U}|$ is $\mathscr{T}$-closed. Moreover, if $\mathcal{U} \in \mathsf{O}(\mathfrak{X}, \leqslant^{\dagger})$, then $|\mathcal{U}|$ is $\mathscr{T}_{\blacksquare}^{-}$-open, cf. Sect. 2.4.1, which implies that $|\mathcal{U}|$ satisfies $\varphi(t, |\mathcal{U}|) \subset \mathrm{int}\,|\mathcal{U}|$ for $t > 0$, cf. Lem. 3.7. These facts combined show that $\mathcal{U} \in \mathsf{O}(\mathfrak{X}, \leqslant^{\dagger})$ implies that $|\mathcal{U}|$ is a $(\mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$-pairwise clopen set and therefore a closed attracting block, i.e. $|\mathcal{U}| \in \mathsf{ABlock}_{\mathscr{C}}(\varphi)$. We have the following commutative diagram:

$$
\begin{tikzcd}
\mathsf{Set}(X) & \mathsf{ABlock}_{\mathscr{C}}(\varphi) \arrow[l, tail, "\supset"'] \\
\mathsf{Set}(\mathfrak{X}) \arrow[u, tail, "|\cdot|"] & \mathsf{O}(\mathfrak{X}, \leqslant^{\dagger}) \arrow[l, tail, "\supset"'] \arrow[u, tail, "|\cdot|"']
\end{tikzcd}
\tag{3.11}
$$

THEOREM 3.19. *Let $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ be a discretization map. A pre-order $(\mathfrak{X}, \leqslant^{\dagger})$ is a Morse pre-order on $\mathfrak{X}$ for $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$ if and only if $|\mathbf{cl}^{\dagger}\xi|$ is $\mathscr{T}$-closed and*

$$\varphi(t, |\xi|) \subset \mathrm{int}\,|\mathbf{cl}^{\dagger}\xi|, \quad \forall t > 0, \tag{3.12}$$

for all $\xi \in \mathfrak{X}$.

PROOF. If $(\mathfrak{X}, \leqslant^{\dagger})$ is a Morse pre-order then for every $\mathcal{U} \in \mathsf{O}(\mathfrak{X}, \leqslant^{\dagger})$, $|\mathcal{U}|$ is $\mathscr{T}$-closed and $\varphi(t, |\mathcal{U}|) \subset \mathrm{int}\,|\mathcal{U}|$ for $t > 0$. Take $\mathcal{U} = \mathbf{cl}^{\dagger}\xi$. Then, $|\mathbf{cl}^{\dagger}\xi|$ is $\mathscr{T}$-closed and $\varphi(t, |\xi|) \subset \varphi(t, |\mathbf{cl}^{\dagger}\xi|) \subset \mathrm{int}\,|\mathbf{cl}^{\dagger}\xi|$ for all $t > 0$.

Conversely, suppose $|\mathbf{cl}^\dagger\xi|$ is $\mathscr{T}$-closed and (3.12) is satisfied. To prove that $(\mathfrak{X}, \leqslant^\dagger)$ is a Morse pre-order we show that it is $\mathscr{T}$-consistent and $\mathscr{T}_\blacksquare^-$-co-consistent with respect to disc. Let $\mathcal{U} \in \mathsf{O}(\mathfrak{X}, \leqslant^\dagger)$. Then, $\mathbf{cl}^\dagger\mathcal{U} = \mathcal{U}$ and

$$|\mathcal{U}| = |\mathbf{cl}^\dagger\mathcal{U}| = |\mathbf{cl}^\dagger \bigcup\{\xi\}| = |\bigcup \mathbf{cl}^\dagger\xi| = \bigcup |\mathbf{cl}^\dagger\xi|,$$

is $\mathscr{T}$-closed. Therefore, $\mathcal{U}$ closed in $(\mathfrak{X}, \leqslant^\dagger)$ implies that $\mathrm{disc}^{-1}\mathcal{U} = |\mathcal{U}|$ is $\mathscr{T}$-closed and thus $\leqslant^\dagger$ is $\mathscr{T}$-consistent with respect to disc. Moreover,

$$\begin{aligned}\varphi(t, |\mathcal{U}|) = \varphi\big(t, \bigcup |\xi|\big) &= \bigcup \varphi(t, |\xi|) \subset \bigcup \mathrm{int}\, |\mathbf{cl}^\dagger\xi| \\ &\subset \mathrm{int} \bigcup |\mathbf{cl}^\dagger\xi| = \mathrm{int}\, |\mathbf{cl}^\dagger\mathcal{U}| = \mathrm{int}\, |\mathcal{U}|, \quad \forall t > 0,\end{aligned}$$

which implies that $|\mathcal{U}|$ is $\mathscr{T}_\blacksquare^-$-open. Therefore, $\mathcal{U}$ closed in $(\mathfrak{X}, \leqslant^\dagger)$ implies that $\mathrm{disc}^{-1}\mathcal{U} = |\mathcal{U}|$ is $\mathscr{T}_\blacksquare^-$-open by (3.6). Recall that $\mathscr{T}_\blacksquare^-$-co-consistency can be characterized as follows: $\mathcal{U}^c \in \mathsf{O}(\mathfrak{X}, \geqslant^\dagger)$, then $|\mathcal{U}|^c = |\mathcal{U}^c|$ is $\mathscr{T}_\blacksquare^-$-closed, which is equivalent to $\mathcal{U} \in \mathsf{O}(\mathfrak{X}, \leqslant^\dagger)$, then $|\mathcal{U}|$ is $\mathscr{T}_\blacksquare^-$-open, cf. Sect. 2.4.1. Using the latter proves that $\leqslant^\dagger$ is $\mathscr{T}_\blacksquare^-$-co-consistent with respect to disc. ☐

REMARK 3.20. The $\mathscr{T}$-consistency of $(\mathfrak{X}, \leqslant^\dagger)$ with respect to disc implies that $\mathrm{cl}|\xi| \subset |\mathbf{cl}^\dagger\xi|$ for all $\xi \in \mathfrak{X}$. In particular, this implies that $\varphi(t, \mathrm{cl}|\xi|) \subset \varphi(t, |\mathbf{cl}^\dagger\xi|) \subset \mathrm{int}\, |\mathbf{cl}^\dagger\mathbf{cl}^\dagger\xi| = \mathrm{int}\, |\mathbf{cl}^\dagger\xi|$, for all $t > 0$ and for all $\xi \in \mathfrak{X}$.

If $\mathbf{\Phi}\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$ is a modal operator such that

$$\varphi\big(t, |\xi|\big) \subset \mathrm{int}\big|\mathbf{\Phi}\xi\big|, \quad \forall t > 0, \quad \text{and} \quad \forall \xi \in \mathfrak{X}. \tag{3.13}$$

then Theorem 3.12 implies that the operator $\mathbf{cl}_{\mathbf{\Phi}} = \mathbf{\Phi}^{+=} = \bigcup_{k\geqslant 0} \mathbf{\Phi}^k\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$ is an antagonistic closure operator for $(X, \mathscr{T}, \mathscr{T}_\blacksquare^-)$, cf. Lem. 3.13. If $(X, \mathscr{T})$ is compact then (3.13) can be weakened to $0 < t \leqslant t_\xi$, cf. Thm. 3.14

3.3.1. Morse tessellations. For a Morse pre-order $(\mathfrak{X}, \leqslant^\dagger)$ the down-sets yield a sublattice $\mathsf{O}(\mathfrak{X}, \leqslant^\dagger)$ of closed attracting blocks. Following the theory in Section 2.5 a Morse pre-order $(\mathfrak{X}, \leqslant^\dagger)$ yields a finite discretization map $\mathrm{dyn}\colon \mathfrak{X} \twoheadrightarrow \mathsf{SC}$ which is defined by combining the formulas in (2.16) and (2.18). The latter is dual to the embedding $\mathsf{O}(\mathsf{SC}) \cong \mathsf{O}(\mathfrak{X}, \leqslant^\dagger) \rightarrowtail \mathsf{Set}(\mathfrak{X})$. The composition

$$X \xrightarrow{\mathrm{disc}} \mathfrak{X} \xrightarrow{\mathrm{dyn}} \mathsf{SC},$$

defines a continuous T_0-discretization of X which is denoted by $\mathrm{tile}\colon X \to \mathsf{SC}$, cf. (2.20) and which defines an SC-grading on X, cf. App. C.1. Compare the latter with the composition $X \xrightarrow{\mathrm{disc}} \mathfrak{X} \xrightarrow{\mathrm{dim}} \mathbb{N}$ which defines an $\mathbb{N}$-grading on X. The diagrams in (1.1) and (2.19) show how dyn is order-preserving and order-reversing with respect to $(\mathfrak{X}, \leqslant)$ and $(\mathfrak{X}, \leqslant_\blacksquare^-)$ respectively, cf. Sect. 2.5. Diagram (1.1) also shows the continuous T_0-discretization maps

$$\mathrm{tile}\colon (X, \mathscr{T}) \to (\mathsf{SC}, \leqslant) \quad \text{and} \quad \mathrm{tile}\colon (X, \mathscr{T}_\blacksquare^-) \to (\mathsf{SC}^*, \geqslant),$$

by factoring through $(\mathfrak{X}, \leqslant)$ and through $(\mathfrak{X}, \leqslant_{\blacksquare}^{-})$ respectively. A Morse pre-order can therefore be thought of as grading of X which, as T_0-discretization, is a continuous map tile in the above sense. A down-set $\mathcal{U}$ in $(\mathsf{SC}, \leqslant)$ is a down-set in $(\mathfrak{X}, \leqslant)$ and an up-set in $(\mathfrak{X}, \leqslant_{\blacksquare}^{-})$ and therefore closed in $(X, \mathscr{T})$ and and open in $(X, \mathscr{T}_{\blacksquare}^{-})$ respectively. By Lemma 3.7 this implies that down-sets $\mathcal{U}$ in $(\mathsf{SC}, \leqslant)$ realize to closed attracting neighborhoods $\mathrm{tile}^{-1}\mathcal{U} \in \mathsf{ABlock}_{\mathscr{C}}(\varphi)$. On the level of classes $\mathcal{S} \in \mathsf{SC}$ the inverse image of tile yields an SC-graded tessellation $(\mathsf{T}, \leqslant)$ with

$$\mathsf{T} := \{T = \mathrm{tile}^{-1}\mathcal{S} \mid \mathcal{S} \in \mathsf{SC}\}, \tag{3.14}$$

such that $\downarrow T$ is $\mathscr{T}$-closed and $\mathscr{T}_{\blacksquare}^{-}$-open, i.e. $\varphi(t, x) \in \mathrm{int} \downarrow T$ for every $x \in T$ and for all tiles T. The latter follows since $\downarrow T = \downarrow \mathrm{tile}^{-1}\mathcal{S} = \mathrm{tile}^{-1} \downarrow \mathcal{S}$ and $\downarrow \mathcal{S}$ is a down-set in SC, cf. [44, Defn. 8]. This motivates the definition:

DEFINITION 3.21 (cf. [44], Cor. 4). An ordered tessellation $(\mathsf{T}, \leqslant)$ of X, cf. Defn. C.1, is called a *Morse tessellation* for ϕ if for every $T \in \mathsf{T}$

(i) $\downarrow T$ is $\mathscr{T}$-closed;
(ii) $\downarrow T$ is $\mathscr{T}_{\blacksquare}^{-}$-open, i.e. $\varphi(t, x) \in \mathrm{int} \downarrow T$, for all $x \in T$ and for all $t > 0$.

The sets $T \in \mathsf{T}$ are called *Morse tiles*.[1]

Conversely, Morse tessellations yield Morse pre-orders and associated space discretizations. Indeed, for a Morse tessellation $(\mathsf{T}, \leqslant)$ we declare the tiles to be the cells in $\mathfrak{X}$ and the partial order is the Morse pre-order on $\mathfrak{X}$. By definition this defines a discretization for both $(X, \mathscr{T})$ and $(X, \mathscr{T}_{\blacksquare}^{-})$. In the next subsection we discuss a more refined reconstruction based on regular closed sets.

REMARK 3.22. One can obviously build larger sets $\mathfrak{X}$ by for example considering $\mathscr{T}$ or $\mathscr{T}_{\blacksquare}^{-}$ closure of the tiles T. One can also define fine structure within the tiles. In the next section we explain a specific reconstruction in the case of regular closed tiles.

Morse tessellations are a defining structure for Morse representations, cf. [44]. The considerations in this subsection explain that Morse tessellations are equivalent to Morse pre-orders which will be the central structure for discussing connection matrix theory in Sect. 4.

3.3.2. Regular closed attracting blocks. This section discusses a special property of closed attracting blocks, cf. (3.3).

THEOREM 3.23. *Let $U \in \mathsf{ABlock}_{\mathscr{C}}(\varphi)$ be a closed attracting block, then U is a regular closed attracting block, i.e. $\mathsf{ABlock}_{\mathscr{C}}(\varphi) = \mathsf{ABlock}_{\mathscr{R}}(\varphi)$, where the latter denotes the lattice of regular closed attracting blocks. Moreover,*

$$U \wedge U' = U \cap U',$$

[1]Equivalently, for every $I \in \mathsf{O}(\mathsf{T}, \leqslant)$, $|I| := \bigcup_{T \in I} T \in \mathsf{ABlock}(\varphi)$, i.e. $\mathsf{T} = \mathsf{T}(\mathsf{N})$ where N is given by $\mathsf{N} = \{|I| \mid I \in \mathsf{O}(\mathsf{T}, \leqslant)\} \subset \mathsf{ABlock}(\varphi)$.

for all $U, U' \in \mathsf{ABlock}_{\mathscr{C}}(\varphi)$.

PROOF. Let $U \subset X$ be a closed attracting block. By definition $\operatorname{cl}\operatorname{int} U \subset U$. Suppose $U \smallsetminus \operatorname{cl}\operatorname{int} U \neq \varnothing$. Let $x \in U \smallsetminus \operatorname{cl}\operatorname{int} U$. Choose $t_n > 0$ with $t_n \to 0$ as $n \to \infty$. Since U is an attracting block we have that $y_n := \varphi(t_n, x) \in \operatorname{int} U$ for all $t_n > 0$. By the continuity $y_n \to x$ as $n \to \infty$ which implies that $x \in \operatorname{cl}\operatorname{int} U$, a contradiction. Therefore $U = \operatorname{cl}\operatorname{int} U$, which proves that U is a regular closed attracting block, cf. [43, 44].

If $U, U' \in \mathsf{ABlock}_{\mathscr{C}}(\varphi)$, then $U \cap U' \in \mathsf{ABlock}_{\mathscr{C}}(\varphi)$ and thus $U \cap U'$ is a closed attracting block and therefore a regular closed attracting block. This implies that $U \cap U' = \operatorname{cl}\operatorname{int}(U \cap U') = U \wedge U'$. $\square$

By Theorem 2.22 we may assume that a Morse pre-order $(\mathfrak{X}, \leqslant^\dagger)$ is induced by a bi-topological CA-discretization $(\mathfrak{X}, \mathbf{cl}, \mathbf{cl}^-_\blacksquare, |\cdot|)$ for $(X, \mathscr{T}, \mathscr{T}^-_\blacksquare)$. Assume without loss of generality that $\mathsf{O}(\mathfrak{X}, \leqslant^\dagger) = \mathscr{C}\mathscr{O}(\mathfrak{X})$ and $\mathbf{cl} = \mathbf{cl}^\dagger$ and $\mathbf{cl}^-_\blacksquare = \mathbf{star}^\dagger$. Moreover, assume that $(\mathfrak{X}, \mathbf{cl}, |\cdot|)$ is Boolean CA-discretization. Let $\mathcal{U} \in \mathscr{C}\mathscr{O}(\mathfrak{X})$, then, since $|\mathcal{U}| \in \mathsf{ABlock}_{\mathscr{R}}(\varphi)$, the set $\mathcal{U}$ is regular closed and $|\mathcal{U} \cap \mathcal{U}'| = |\mathcal{U}| \cap |\mathcal{U}'| = |\mathcal{U}| \wedge |\mathcal{U}'| = |\mathcal{U} \wedge \mathcal{U}'|$, which yields the following regular closed analogue of (3.11):

$$
\begin{tikzcd}
\mathscr{R}(X) & \mathsf{ABlock}_{\mathscr{R}}(\varphi) \arrow[l, tail, "\supset"'] \\
\mathscr{R}(\mathfrak{X}) \arrow[u, tail, "|\cdot|"] & \mathsf{O}(\mathfrak{X}, \leqslant^\dagger) \arrow[l, tail, "\supset"'] \arrow[u, tail, "|\cdot|"']
\end{tikzcd}
\tag{3.15}
$$

The down-sets for a Morse pre-order yield a sublattice of $\mathsf{ABlock}_{\mathscr{C}}(\varphi)$. Conversely, for a finite sublattice $\mathsf{N} \subset \mathsf{ABlock}_{\mathscr{C}}(\varphi)$ we can construct a Boolean CA-discretization, cf. Sect. 3.4.2 and Rem. 3.31. Morse pre-orders provide an important reduction of discretization data as is explained in the next section.

3.4. Condensed Morse pre-orders

In this section we assume that $(\mathfrak{X}, \mathbf{cl}, |\cdot|)$ is Boolean CA-discretization of $(X, \mathscr{T})$. The fact that closed attracting blocks are regular closed sets, cf. Thm. 3.23, is a crucial property for reducing the data structures in the theory of Morse pre-orders. Such a discretization will be referred to as a *condensed Morse pre-order* .

3.4.1. Pre-orders on top cells. The bottom embedding $\mathsf{O}(\mathfrak{X}, \leqslant^\dagger) \rightarrowtail \mathscr{R}(\mathfrak{X})$ in (3.15) is inclusion since $\mathcal{U} \cap \mathcal{U}' = \mathcal{U} \wedge \mathcal{U}'$, which follows from the fact that the evaluation map $|\cdot| \colon \mathscr{R}(\mathfrak{X}) \rightarrowtail \mathscr{R}(X)$ is a homomorphism and $(\mathfrak{X}, \mathbf{cl}, |\cdot|)$ is Boolean. Since $\mathscr{R}(\mathfrak{X}) \cong \mathsf{Set}(\mathfrak{X}^\top)$, cf. Prop. 2.26, we can dualize the homomorphism

$$\mathsf{O}(\mathsf{SC}, \leqslant) \xrightarrow{\cong} \mathsf{O}(\mathfrak{X}, \leqslant^\dagger) \rightarrowtail \mathscr{R}(\mathfrak{X}) \xrightarrow{\cong} \mathsf{Set}(\mathfrak{X}^\top),$$

which yields the surjection $\pi \colon \mathfrak{X}^\top \twoheadrightarrow (\mathsf{SC}, \leqslant)$, where $\mathfrak{X}^\top$ is unordered and where $\mathsf{SC} \cong \mathsf{J}\big(\mathsf{O}(\mathfrak{X}, \leqslant^\dagger)\big)$.

DEFINITION 3.24. The induced pre-order $\leqslant^\top$ on $\mathfrak{X}^\top$, defined by

$$\xi \leqslant^\top \xi' \quad \text{if and only if} \quad \pi(\xi) \leqslant \pi(\xi'),$$

is called a called a *condensed Morse pre-order* for $\leqslant^\dagger$.[2]

By construction we have that $\mathsf{O}(\mathfrak{X}, \leqslant^\dagger) \cong \mathsf{O}(\mathfrak{X}^\top, \leqslant^\top) \cong \mathsf{O}(\mathsf{SC}, \leqslant)$. The associated closure operator on $\mathsf{Set}(\mathfrak{X}^\top)$ is denoted by $\mathbf{cl}^\top$. By Proposition 2.26, $\mathcal{U}^\top \in \mathsf{O}(\mathfrak{X}^\top, \leqslant^\top)$ implies $\mathbf{cl}\,\mathcal{U}^\top \in \mathsf{O}(\mathfrak{X}, \leqslant^\dagger)$ and thus, since $(\mathfrak{X}, \mathbf{cl}, |\cdot|)$ is Boolean, $\mathrm{cl}\,|\mathcal{U}^\top| = |\mathbf{cl}\,\mathcal{U}^\top| \in \mathsf{ABlock}_{\mathscr{R}}(\varphi)$. Consequently $\varphi(t, |\mathbf{cl}\,\mathcal{U}^\top|) \subset \mathrm{int}\,|\mathbf{cl}\,\mathcal{U}^\top|$ for all $t > 0$. Let $\mathcal{U}^\top = \mathbf{cl}^\top\xi$, with $\xi \in \mathfrak{X}^\top$. Then, $\varphi(t, \|\xi\|) \subset \varphi(t, |\mathbf{cl}\,\mathbf{cl}^\top\xi|) \subset \mathrm{int}\,|\mathbf{cl}\,\mathbf{cl}^\top\xi| = \mathrm{int}\,\mathrm{cl}|\mathbf{cl}^\top\xi| = \mathrm{int}\,\|\mathbf{cl}^\top\xi\|$ for all $t > 0$. The latter is a condition on only the top cells. We show below that any pre-order $(\mathfrak{X}^\top, \leqslant^\top)$ satisfying the latter is a condensed Morse pre-order induced by a Morse pre-order.

THEOREM 3.25. *A pre-order $(\mathfrak{X}^\top, \leqslant^\top)$ is a condensed Morse pre-order for $\leqslant^\dagger$ if and only if*

$$\varphi(t, \|\xi\|) \subset \mathrm{int}\|\mathbf{cl}^\top\xi\|, \quad \forall t > 0. \tag{3.16}$$

Condition (3.16) is a characterization of condensed Morse pre-orders and can be used as alternative definition of condensed Morse pre-order .

REMARK 3.26. If $(X, \mathscr{T})$ is a compact topological space then the criterion in (3.16) is equivalent to the condition: for every $\xi \in \mathfrak{X}^\top$ there exists a $t_\xi > 0$, such that $\varphi(t, \|\xi\|) \subset \mathrm{int}\|\mathbf{cl}^\top\xi\|$ for all $0 < t \leqslant t_\xi$.

REMARK 3.27. For a binary relation $\phi \subset \mathfrak{X}^\top \times \mathfrak{X}^\top$ the transitive, reflexive closure $\phi^{+=}$ defines a pre-order on $\mathfrak{X}^\top$. If (3.16) is satisfied with $\mathbf{cl}^\top{}_\phi = (\phi^{+=})^{-1}$, then ϕ will also be referred to as a condensed Morse pre-order for $\leqslant^\dagger$. The definition of condensed Morse pre-order is reminiscent of the notion of weak outer approximation in relation to the commutative diagram in (3.15), cf. [40], [43, Defn. 3.7], [44].

The Boolean algebra $\mathsf{Set}(\mathfrak{X}^\top)$ is a sublattice in $\mathsf{Set}(\mathfrak{X})$. The embedding does not preserve the top element and the inclusion is not therefore Boolean. The closure operator $\mathrm{cl}\colon \mathsf{Set}(\mathfrak{X}^\top) \to \mathsf{Set}(\mathfrak{X})$, given by $\mathcal{U} \mapsto \mathbf{cl}\,\mathcal{U} \in \mathsf{O}(\mathfrak{X}, \leqslant)$, is additive but not a lattice homomorphism in general. Since $i\colon \mathfrak{X}^\top \hookrightarrow (\mathfrak{X}, \leqslant)$, with $\mathfrak{X}^\top$ unordered, is an order-embedding[3] Birkhoff duality yields the surjective lattice homomorphism $j\colon \mathsf{O}(\mathfrak{X}, \leqslant) \twoheadrightarrow \mathsf{Set}(\mathfrak{X}^\top)$ given by $\mathcal{U} \mapsto \mathcal{U}^\top := \mathcal{U} \cap \mathfrak{X}^\top$. Schematically we pose the

[2] Since $\mathsf{O}(\mathfrak{X}, \leqslant^\dagger) \cong \mathsf{O}(\mathsf{SC}, \leqslant) \cong \mathsf{O}(\mathfrak{X}^\top, \leqslant^\top)$ it follows that a pre-order $\leqslant^\top$ is the restriction of $\leqslant^\dagger$ to the top cells $\mathfrak{X}^\top$, cf. Proof of Thm. 3.25.

[3] The top cells $\mathfrak{X}^\top$ form an anti-chain in $(\mathfrak{X}, \leqslant)$, cf. Sect. 2.6.1.

following lifting diagram:

$$
\begin{tikzcd}
 & \mathsf{O}(\mathfrak{X},\leqslant) \arrow[d, two heads, "j"] \\
\mathsf{O}(\mathfrak{X}^\top,\leqslant^\top) \arrow[r, tail, "\mathrm{id}"] \arrow[ur, dotted, "\mathbf{cl}"] & \mathsf{Set}(\mathfrak{X}^\top)
\end{tikzcd}
\tag{3.17}
$$

Theorem 3.28 below shows that the identity map can be lifted as closure.

THEOREM 3.28. *Suppose $(\mathfrak{X}^\top,\leqslant^\top)$ is a pre-order that satisfies (3.16). Then, the restriction $\mathbf{cl}\colon \mathsf{O}(\mathfrak{X}^\top,\leqslant^\top) \rightarrowtail \mathsf{O}(\mathfrak{X},\leqslant)$ is an injective lattice homomorphism with $j\circ\mathbf{cl} =$ id.*

PROOF. By definition $\mathcal{U}^\top = \bigcup_{\xi\in\mathcal{U}^\top}\{\xi\}$ and since $\|\cdot\|\colon \mathsf{Set}(\mathfrak{X}^\top)\to\mathscr{R}(X)$ is an injective Boolean homomorphism, cf. Prop. 2.27, we have that $\|\mathcal{U}^\top\| = \bigcup_{\xi\in\mathcal{U}^\top}\|\xi\|$. In combination with (3.16), the additivity of $\mathbf{cl}^\top$ and the fact that $\mathcal{U}^\top\in\mathsf{O}(\mathfrak{X}^\top,\leqslant^\top)$ we conclude

$$
\begin{aligned}
\varphi\big(t,\|\mathcal{U}^\top\|\big) &= \bigcup_{\xi\in\mathcal{U}^\top}\varphi\big(t,\|\xi\|\big) \subset \bigcup_{\xi\in\mathcal{U}^\top}\Big(\mathrm{int}\big\|\mathbf{cl}^\top\xi\big\|\Big) \subset \mathrm{int}\Big(\bigcup_{\xi\in\mathcal{U}^\top}\big\|\mathbf{cl}^\top\xi\big\|\Big)\\
&= \mathrm{int}\Big\|\bigcup_{\xi\in\mathcal{U}^\top}\mathbf{cl}^\top\xi\Big\| = \mathrm{int}\big\|\mathbf{cl}^\top\mathcal{U}^\top\big\| \subset \mathrm{int}\,\|\mathcal{U}^\top\|,\quad \forall t>0,
\end{aligned}
$$

which proves that $\|\mathcal{U}^\top\|$ is an attracting block for ϕ. Since $\mathsf{O}(\mathfrak{X}^\top,\leqslant^\top)$ and $\mathsf{ABlock}_{\mathscr{R}}(\varphi)$ are sublattices of $\mathsf{Set}(\mathfrak{X}^\top)$ and $\mathscr{R}(X)$ respectively, and since $\|\cdot\|\colon \mathsf{Set}(\mathfrak{X}^\top)\rightarrowtail\mathscr{R}(X)$ is an injective Boolean homomorphism the evaluation map $\|\cdot\|\colon \mathsf{O}(\mathfrak{X}^\top,\leqslant^\top)\rightarrowtail \mathsf{ABlock}_{\mathscr{R}}(\varphi)$ is a injective lattice homomorphism. In particular we conclude that $|\mathbf{cl}\,\mathcal{U}^\top|\in\mathsf{ABlock}_{\mathscr{R}}(\varphi)$ and $\mathbf{cl}\,\mathcal{U}^\top\in\mathsf{O}(\mathfrak{X},\leqslant)$.

To show that the restriction of $\mathbf{cl}$ is a homomorphism it remains to check that the unit and intersection are preserved. By definition $\mathbf{cl}\,\mathfrak{X}^\top = \mathfrak{X}$ which proves that the unit is preserved. By Proposition 2.26 we have that $\mathbf{cl}\,\mathcal{U}^\top\in\mathscr{R}(\mathfrak{X})$ and by Theorem 3.23, $|\mathbf{cl}\,\mathcal{U}^\top|\cap|\mathbf{cl}\,\mathcal{U}'^\top| = |\mathbf{cl}\,\mathcal{U}^\top|\wedge|\mathbf{cl}\,\mathcal{U}'^\top|$. This implies,

$$
\begin{aligned}
|\mathbf{cl}(\mathcal{U}^\top\cap\mathcal{U}'^\top)| &= \mathrm{cl}|\mathcal{U}^\top\cap\mathcal{U}'^\top| = \|\mathcal{U}^\top\cap\mathcal{U}'^\top\| = \|\mathcal{U}^\top\|\wedge\|\mathcal{U}'^\top\| = \|\mathcal{U}^\top\|\cap\|\mathcal{U}'^\top\|\\
&= |\mathbf{cl}\,\mathcal{U}^\top|\cap|\mathbf{cl}\,\mathcal{U}'^\top| = |\mathbf{cl}\,\mathcal{U}^\top\cap\mathbf{cl}\,\mathcal{U}'^\top|,
\end{aligned}
$$

where we use Proposition 2.27 to conclude that $\|\mathcal{U}^\top\cap\mathcal{U}'^\top\| = \|\mathcal{U}^\top\|\wedge\|\mathcal{U}'^\top\|$. The fact that $|\cdot|$ is injective yields $\mathbf{cl}(\mathcal{U}^\top\cap\mathcal{U}'^\top) = \mathbf{cl}\,\mathcal{U}^\top\cap\mathbf{cl}\,\mathcal{U}'^\top$, which completes the proof. $\square$

PROOF OF THM. 3.25. The direction that a condensed Morse pre-order satisfies (3.16) is given above. It remains to show that (3.16) yields a Morse pre-order. Suppose (3.16) is satisfied. Then, by Theorem 3.28, $\mathbf{cl}\colon \mathsf{O}(\mathfrak{X}^\top,\leqslant^\top)\to\mathsf{O}(\mathfrak{X},\leqslant)$ provided an embedding sublattice. From (2.15)-(2.17) we obtain a pre-order $(\mathfrak{X},\leqslant^\dagger)$

such that the range of the above closure is the lattice $\mathsf{O}(\mathfrak{X}, \leqslant^\dagger)$:

$$\begin{tikzcd}
\mathsf{O}(\mathfrak{X}, \leqslant^\dagger) \arrow[dr, tail, "\subset"'] & & \mathsf{O}(\mathfrak{X}^\top, \leqslant^\top) \arrow[ll, "\mathbf{cl}", "\cong"'] \arrow[dl, tail, "\mathbf{cl}"] \\
& \mathsf{O}(\mathfrak{X}, \leqslant) &
\end{tikzcd}$$

The pre-order $(\mathfrak{X}, \leqslant^\dagger)$ is the desired Morse pre-order that induces $(\mathfrak{X}^\top, \leqslant^\top)$. If we choose $\leqslant_{\blacksquare}^- = \geqslant^\dagger$ we obtain $\leqslant^\dagger$ as antagonistic pre-order for $(\mathfrak{X}, \mathbf{cl}, \mathrm{cl}_{\blacksquare}^-, |\cdot|)$, cf. Thm. 2.22. $\square$

The novelty of the above construction is that the (injective) composition

$$\mathsf{O}(\mathsf{SC}, \leqslant) \xrightarrow{\cong} \mathsf{O}(\mathfrak{X}^\top, \leqslant^\top) \xrightarrow{\mathbf{cl}} \mathsf{O}(\mathfrak{X}, \leqslant) \xrightarrow{\subset} \mathsf{Set}(\mathfrak{X})$$

dualizes to the finite discretization

$$\mathrm{dyn}\colon (\mathfrak{X}, \leqslant) \twoheadrightarrow (\mathsf{SC}, \leqslant), \tag{3.18}$$

which is defined in Section 3.3.1 and is given by the formulas in (2.16) and (2.18). The finite discretization dyn recovers the Morse pre-order $\leqslant^\dagger$ via $\xi \leqslant^\dagger \xi'$ if and only if $\mathrm{dyn}\,\xi \leqslant \mathrm{dyn}\,\xi'$, and $\leqslant^\top$ is the restriction of $\leqslant^\dagger$ to $\mathfrak{X}^\top$. The advantage of using regular closed sets is that the Morse pre-order is completely determined by the restriction $(\mathfrak{X}^\top, \leqslant^\top)$ which is a much smaller data structure in general and bypasses the topologies given by $\leqslant$ and $\leqslant_{\blacksquare}^-$. The following result gives a formula for determining dyn in terms of $\leqslant^\top$:

THEOREM 3.29. *Suppose $(\mathfrak{X}^\top, \leqslant^\top)$ is a pre-order that satisfies (3.16). The finite discretization $\mathrm{dyn}\colon (\mathfrak{X}, \leqslant) \twoheadrightarrow (\mathsf{SC}, \leqslant)$ is given by*

$$\xi \mapsto \mathrm{dyn}(\xi) = \min_{\mathsf{SC}} \left\{ \left[\eta^\top\right] \mid \eta^\top \in \mathbf{star}\,\xi \cap \mathfrak{X}^\top \right\}, \tag{3.19}$$

where $\left[\eta^\top\right] \in \mathsf{SC}$ is the partial equivalence class in $(\mathfrak{X}^\top, \leqslant^\top)$ containing $\eta^\top$.

PROOF. Consider the commutative diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, tail, "\mathrm{dyn}"] \arrow[d, "\iota_{\mathfrak{X}}"'] & \mathsf{SC} \arrow[d, "\iota_{\mathsf{SC}}"] \\
\mathsf{J}(\mathsf{O}(\mathfrak{X}, \leqslant)) \arrow[r, tail, "\mathsf{J}(\mathbf{cl})"] & \mathsf{J}(\mathsf{O}(\mathsf{SC}))
\end{tikzcd} \tag{3.20}$$

the maps $\iota_{\mathfrak{X}}$ and ι_{SC} are given by $\xi \xmapsto{\iota_{\mathfrak{X}}} \downarrow\xi$ and $[\xi^\top] \xmapsto{\iota_{\mathsf{SC}}} \downarrow[\xi^\top]$, and $\mathbf{cl}\colon \mathsf{O}(\mathsf{SC}) \rightarrowtail \mathsf{O}(\mathfrak{X}, \leqslant)$. By the commutativity we have that $\mathrm{dyn} = \iota_{\mathsf{SC}}^{-1} \circ \mathsf{J}(\mathbf{cl}) \circ \iota_{\mathfrak{X}}$, and $\mathsf{J}(\mathbf{cl})(\mathcal{U}) = \min \mathbf{cl}^{-1}(\uparrow\mathcal{U}) \in \mathsf{J}(\mathsf{O}(\mathsf{SC}))$, $\mathcal{U} \in \mathsf{J}(\mathsf{O}(\mathfrak{X}, \leqslant))$, cf. Thm. B.2. Recall that $\mathbf{cl}^{-1}(\mathcal{U}) = \{\mathcal{U}^\top \in \mathsf{O}(\mathsf{SC}) \mid \mathbf{cl}\,\mathcal{U}^\top = \mathcal{U}\}$. Note that in $\mathsf{O}(\mathfrak{X}, \leqslant)$ the up-set $\uparrow\iota_{\mathfrak{X}}(\xi)$ is the set of closed subsets in $\mathfrak{X}$ that contain $\iota_{\mathfrak{X}}(\xi) = \downarrow\xi$. By definition $\mathbf{cl}^{-1}(\uparrow\iota_{\mathfrak{X}}(\xi))$ are all $\mathcal{U}^\top \in \mathsf{O}(\mathsf{SC})$ such that $\mathbf{cl}\,\mathcal{U}^\top = \mathcal{U}$ for some $\mathcal{U} \in \mathsf{O}(\mathfrak{X}, \leqslant)$ with $\downarrow\xi \subset \mathcal{U}$. The latter is equivalent to $\xi \in \mathcal{U}$. Since $\mathbf{cl}\,\mathcal{U}^\top \in \mathsf{O}(\mathsf{SC})$ this implies that $\mathbf{cl}^{-1}(\uparrow\iota_{\mathfrak{X}}(\xi)) = \{\mathcal{U}^\top \in \mathsf{O}(\mathsf{SC}) \mid \xi \in \mathbf{cl}\,\mathcal{U}^\top\}$. Since join-irreducible elements generate all elements in a finite distributive lattice we have that $\mathbf{cl}^{-1}\big(\uparrow\iota_{\mathfrak{X}}(\xi)\big) = \big\{\mathcal{U}^{\top} \in \mathsf{J}(\mathsf{O}(\mathsf{SC})) \mid \xi \in \mathbf{cl}\,\mathcal{U}^{\top}\big\}$ and thus

$$\mathsf{J}(\mathbf{cl})(\iota_{\mathfrak{X}}(\xi)) = \min\big\{\mathcal{U}^{\top} \in \mathsf{J}(\mathsf{O}(\mathsf{SC})) \mid \xi \in \mathbf{cl}\,\mathcal{U}^{\top}\big\},$$

which is attained by a unique $\widehat{\mathcal{U}}^{\top} \in \mathsf{J}(\mathsf{O}(\mathsf{SC}))$. Recall that $\mathcal{U}^{\top} \in \mathsf{J}(\mathsf{O}(\mathsf{SC}))$ if and only if $\mathcal{U}^{\top} = \downarrow\mathcal{S}$, $\mathcal{S} = [\xi^{\top}]$ for some $\xi^{\top} \in \mathfrak{X}^{\top}$. Therefore, $\xi \in \mathbf{cl}\,\mathcal{U}^{\top}$ if and only if $\xi \in \mathbf{cl}\,\eta^{\top}$ for some $\eta^{\top} \in \downarrow[\xi^{\top}]$. By duality $\xi \in \mathbf{cl}\,\eta^{\top}$ if and only if $\eta^{\top} \in \mathbf{star}\,\xi \cap \mathfrak{X}^{\top}$. Consider the set

$$\big\{[\eta^{\top}] \in \mathsf{SC} \mid \eta^{\top} \in \mathbf{star}\,\xi \cap \mathfrak{X}^{\top}\big\}.$$

Let $[\eta^{\top}]$ and $[\tilde{\eta}^{\top}]$ be minimal. Then, $\downarrow[\eta^{\top}] \subset \widehat{\mathcal{U}}^{\top}$ and $\downarrow[\tilde{\eta}^{\top}] \subset \widehat{\mathcal{U}}^{\top}$, which implies that $[\eta^{\top}] = [\tilde{\eta}^{\top}]$, and $\downarrow[\eta^{\top}] = \widehat{\mathcal{U}}^{\top}$. $\square$

REMARK 3.30. For condensed Morse pre-orders the commutative diagram in (3.15) is replaced by

$$\begin{tikzcd} \mathscr{R}(X) & \mathsf{ABlock}_{\mathscr{R}}(\varphi) \arrow[l, hook', "\supset"'] \\ \mathsf{Set}(\mathfrak{X}^{\top}) \arrow[u, "\|\cdot\|"] & \mathsf{O}(\mathfrak{X}^{\top}, \leqslant^{\top}) \arrow[l, hook', "\supset"'] \arrow[u, "\|\cdot\|"'] \end{tikzcd} \tag{3.21}$$

Condensed Morse pre-orders give rise to regular closed Morse tessellations.

3.4.2. Regular closed tessellations. A condensed Morse pre-order yields a Morse pre-order. The range of the injective lattice homomorphism $\mathbf{cl}\colon \mathsf{O}(\mathfrak{X}^{\top}, \leqslant^{\top}) \to \mathsf{O}(\mathfrak{X}, \leqslant)$ can be expressed as $\mathsf{O}(\mathfrak{X}, \leqslant^{\dagger})$ via a pre-order $(\mathfrak{X}, \leqslant^{\dagger})$ — a Morse pre-order. For $\mathcal{U}^{\top} \in \mathsf{O}(\mathfrak{X}^{\top}, \leqslant^{\top})$ we can give a representation of SC in terms of regular closed tiles. By construction $\|\mathcal{U}^{\top}\| \in \mathsf{ABlock}_{\mathscr{R}}(\varphi)$ and we denote the associated sublattice of regular closed attracting blocks by $\mathsf{N} \subset \mathsf{ABlock}_{\mathscr{R}}(\varphi)$. Then, $\mathsf{SC} \cong \mathsf{J}(\mathsf{N}) \cong \mathsf{T}$, where $T = \|\mathcal{U}^{\top}\| - \|\mathcal{U}^{\top\blacktriangleleft}\| := \|\mathcal{U}^{\top}\| \wedge \|\mathcal{U}^{\top\blacktriangleleft}\|^{\#} \in \mathscr{R}(X)$, cf. [44]. From [44, Lem. 23] we have that

$$\begin{aligned} \|\mathcal{U}^{\top}\| - \|\mathcal{U}^{\top\blacktriangleleft}\| &= \mathrm{cl}\,\big(\|\mathcal{U}^{\top}\| \smallsetminus \|\mathcal{U}^{\top\blacktriangleleft}\|\big) = \mathrm{cl}\Big(|\mathbf{cl}\,\mathcal{U}^{\top}| \smallsetminus |\mathbf{cl}\,\mathcal{U}^{\top\blacktriangleleft}|\Big) \\ &= \mathrm{cl}\Big(\big|\mathbf{cl}\,\mathcal{U}^{\top} \smallsetminus \mathbf{cl}\,\mathcal{U}^{\top\blacktriangleleft}\big|\Big) = \Big|\mathbf{cl}\big(\mathbf{cl}\,\mathcal{U}^{\top} \smallsetminus \mathbf{cl}\,\mathcal{U}^{\top\blacktriangleleft}\big)\Big| \\ &= \big|\mathbf{cl}\,\mathcal{U}^{\top} - \mathbf{cl}\,\mathcal{U}^{\top\blacktriangleleft}\big|, \end{aligned}$$

which shows that the regular closed tiles are closure of the Morse tiles obtained in (2.18) and (3.14). The poset $(\mathsf{T}, \leqslant)$, which is isomorphic to $(\mathsf{SC}, \leqslant)$, is an example of a *regular closed Morse tessellation*. The definition of a regular closed Morse tessellation is similar to Definition 3.21: the tiles are regular closed and Condition (i) is redundant. Given a regular closed Morse tessellation we can reconstruct a Morse pre-order. If we start with a regular closed Morse tessellation $(\mathsf{T}, \leqslant)$, then the Morse tiles $T \in \mathsf{T}$ generate a subalgebra of regular closed sets $\mathscr{R}_0(X)$ which in turn generates a finite subalgebra of $\mathsf{Set}(X)$ represented by $\mathsf{Set}(\mathfrak{X})$ for some finite set $\mathfrak{X}$, cf. Prop. 2.28. The elements in $\mathfrak{X}$ are again denoted by ξ and their *realization* in X is denoted by $|\xi|$. This way we obtain a discrete space $\mathfrak{X}$ with two pre-orders: (i) the face pre-order $\leqslant$ defined by $\xi \leqslant \xi'$ if and only if $|\xi| \subseteq \mathrm{cl}|\xi'|$, for which closed sets in $(\mathfrak{X}, \leqslant)$ correspond to closed sets in the topological space $(X, \mathscr{T})$ and (ii) the Morse pre-order $\leqslant^\dagger$, derived from the Morse tessellation, defined by $\xi \leqslant^\dagger \xi'$ if and only if $\varphi\big(t, |\xi|\big) \in \mathrm{int} \downarrow T$ for all $t > 0$ for some $T \supset |\xi'|$. Closed sets in $(\mathfrak{X}, \leqslant^\dagger)$ correspond to regular closed attracting blocks for the semi-flow φ, and the partial equivalence classes of $\leqslant^\dagger$ retrieve the Morse tessellation partial order. Summarizing, a regular closed Morse tessellation gives rise to a bi-topological CA-discretization $(\mathfrak{X}, \leqslant, \leqslant_{\blacksquare}^{-}, |\cdot|)$ where $\leqslant_{\blacksquare}^{-} = \geqslant^\dagger$.

REMARK 3.31. If we choose an arbitrary finite sublattice $\mathsf{N} \subset \mathsf{ABlock}_{\mathscr{R}}(\varphi)$, then N is a sublattice of some finite subalgebra $\mathscr{R}_0(X) \subset \mathscr{R}(X)$. This induces a Boolean CA-discretization by Proposition 2.28.

3.5. Beyond semi-flows

In this chapter the focus of applying bi-topological techniques is restricted to semi-flows. However, most of the ideas and methods apply to a much wider class of dynamical systems. In this section we outline some of these extensions and how these fits into the theory of this chapter.

A *relational semi-flow* $\phi = \{\phi^t\}_{t \in \mathbb{T}^+}$ is a family of binary relations $\phi^t \subset X \times X$ on a point set X parametrized by (time) $t \in \mathbb{T}^+$ such that

(i) $\phi^0 = \mathrm{id}$ on X;
(ii) $\phi^s \circ \phi^t = \phi^{s+t}$ for all[4] $s, t \in \mathbb{T}^+$.

The time space $\mathbb{T}^+$ is either $\mathbb{Z}^+$ or $\mathbb{R}^+$. For negative time we define ϕ^{-t} to be the *opposite relation*, i.e. $\phi^{-t} = \big\{(x, y) \in X \times X \mid (y, x) \in \phi^t\big\}$, cf. App. A. Therefore Axiom (ii) is equivalent to

(ii)$'$ $\phi^s \circ \phi^t = \phi^{s+t}$ for all $s \cdot t \geqslant 0$ with $s, t \in \mathbb{T}$,

where $\mathbb{T}$ is either $\mathbb{Z}$ or $\mathbb{R}$. In Appendix A we discuss additional properties of binary relations. If $\varphi(t, x) := \phi^t(x)$ defines a continuous map $\mathbb{T}^+ \times X \to X$ then, if $\mathbb{T}^+ = \mathbb{R}^+$, φ is called a *continuous semi-flow* on X which is the main point of focus in this text. If $\mathbb{T}^+ = \mathbb{Z}^+$, ϕ is a called an *iterated continuous map*. In this case it suffices to only consider the map $f := \phi^1$ since higher iterates are found via composition. Backward images define ϕ^t for negative times. Most considerations in this chapter are valid for relational semi-flows with $\mathbb{T}$ either discrete or continuous time. In particular the techniques carry over to iterated maps. We will indicate in which situations continuity will be required.

On the complete and atomic Boolean algebra $\mathsf{Set}(X)$ there is a natural duality between binary relations and completely additive modal operators, cf. App. A. Let Φ be a modal operator on $\mathsf{Set}(X)$. Recall from Sect. 2.2 that specialization relation is given by $(x, x') \in \phi$ if and only if $x \in \Phi\{x'\}$ and the operator $\Phi = \phi^{-1}$ is defined

[4] For composition of relations and other properties cf. App. A.

via (2.4). We apply this principle to a relational semi-flow by setting

$$(x, x') \in \phi^{-t} \quad \text{if and only if} \quad x \in \phi^t\{x'\}, \quad t \in \mathbb{T}.$$

If we coarsen the relations by discarding time we obtain the relation

$$(x, x') \in \phi^{-t}, \text{ for some } \quad t > 0 \quad \text{if and only if} \quad x \in \Gamma^+\{x'\} := \bigcup_{t>0} \phi^t\{x'\},$$

which is remeniscent of the operator Γ^+ defined in Section 3.1.1. A similar definition can be made for Γ^-. Via Γ^+ and Γ^- one can define associated Alexandrov topologies $\mathscr{T}^+$ and $\mathscr{T}^-$ respectively. It makes sense to define finer topologies via appropriately defined modal operators. For $\tau \in \mathbb{T}^+$ define the topology $\mathscr{T}_\tau$ by declaring the sets $U \subset X$ such that $\phi^\tau U \subset U$ to be closed. In particular we have that

$$\mathscr{T}^+ \subset \mathscr{T}^+_\tau \subset \mathscr{T}_\tau, \quad \tau > 0,$$

and the same for $\tau < 0$. All these topologies are Alexandrov. The specialization relation for $\mathscr{T}_\tau$ is given by

$$(x, x') \in \phi^{-\tau} \quad \text{if and only if} \quad y \in \phi^\tau\{x\}.$$

The flow topologies discussed in Section 3.1.2 are not Alexandrov in general and the associated duality is more involved. For example the block-flow topology $\mathscr{T}_{\blacksquare}^-$ for a relational semi-flow is defined by considering the modal operator Γ^- as defined above in the setting of relational semi-flows. Define the modal operator $\Phi_{\blacksquare}^- := \Gamma^- \mathrm{cl}$ for the topology $\mathscr{T}_{\blacksquare}^-$. Another interesting modal operator to consider is defined as: $\Phi_{\blacksquare}^\tau = \phi^\tau \mathrm{cl}$.

Finally, even though Theorem 3.23 does not hold in general for relational semi-flows one can also study regular closed attracting blocks for relational semi-flows, cf. [44].

CHAPTER 4

Algebraization of dynamics

In this section, we elaborate on the third theme of this text: augmentation of Morse pre-orders with algebraic topological data in order to characterize invariance of the dynamics, i.e. the algebraization of dynamics. In particular, we use techniques from algebraic topology in a way in that enables a computational theory. The starting point is a discretization. A Morse pre-order $(\mathfrak{X}, \leqslant^{\dagger})$ is the choice of a pre-order such that the discretization map $\mathrm{disc}\colon (X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-}) \twoheadrightarrow (\mathfrak{X}, \leqslant^{\dagger})$ is both $\mathscr{T}$-consistent and $\mathscr{T}_{\blacksquare}^{-}$-co-consistent. In particular, the composed maps

$$\begin{tikzcd}
(X, \mathscr{T}) \arrow[r, two heads, "\mathrm{disc}"] \arrow[rr, bend right, two heads, "\mathrm{tile}"'] & (\mathfrak{X}, \leqslant^{\dagger}) \arrow[r, two heads, "\mathrm{dyn}"] & (\mathsf{SC}, \leqslant), & (X, \mathscr{T}_{\blacksquare}^{-}) \arrow[r, two heads, "\mathrm{disc}"] \arrow[rr, bend right, two heads, "\mathrm{tile}"'] & (\mathfrak{X}, \geqslant^{\dagger}) \arrow[r, two heads, "\mathrm{dyn}"] & (\mathsf{SC}^{*}, \geqslant)
\end{tikzcd}$$

are continuous T_0-discretizations, denoted by tile, which define SC-gradings on X, cf. App. C.1. We explain how factorized gradings can be used to discretize algebraic topological invariants of topological spaces. We apply these methods in the context of space and flow topologies. Recall that the first theme was linking topology and dynamics by formulating dynamics as a topology. It is worthwhile to then ask of the reverse direction: what happens when topology is analyzed as dynamics? The beginning of this chapter explores this direction, leading to a construction we entitle *tessellar homology*, which, in contradistinction to cellular homology, uses general tiles instead of CW-cells.

4.1. Cartan-Eilenberg systems

The notion of a Cartan-Eilenberg system over a (countable) total order was first introduced in [11].This notion is generalized to arbitrary total orders in [35] and to arbtrary posets in [48] and [63]. Here we use this concept for finite distributive lattices, cf. [63]. To some extend Cartan-Eilenberg systems may be regarded as a type of generalized homology theory. These systems provide the right data structure for considering algebraic topological invariance in combination with filterings and discretizations of a space.

4.1.1. Cartan-Eilenberg systems over finite distributive lattices. Birkhoff's representation theorem, cf. Thm. B.2, yields that every finite distributive lattice can be represented as the down-set lattice $\mathsf{O}(\mathsf{P})$ for some finite poset $(\mathsf{P}, \leqslant)$. Regard $\mathsf{O}(\mathsf{P})$ as small (thin) category where the objects are the elements in the lattice and the order relations $\alpha \leqslant \beta$ (i.e. $\alpha \subset \beta$) account for the morphisms, or arrows, i.e.

$\alpha \leqslant \beta$ yields the arrow $\alpha \to \beta$. The *arrow category* of $\mathsf{O}(\mathsf{P})$ consists of pairs (α, β), with $\alpha \leqslant \beta$, and unique morphisms $(\alpha, \beta) \to (\gamma, \delta)$ for $\alpha \leqslant \gamma$ and $\beta \leqslant \delta$, and is denoted by $\mathsf{O}(\mathsf{P})^{\mathbf{2}}$ and corresponds to commutative diagrams in $\mathsf{O}(\mathsf{P})$. Following [35] we consider the covariant functors p_0, p_1 and p_2 given by $(\alpha, \beta, \gamma) \xmapsto{\mathsf{p}_0} (\alpha, \beta)$, $(\alpha, \beta, \gamma) \xmapsto{\mathsf{p}_1} (\alpha, \gamma)$ and $(\alpha, \beta, \gamma) \xmapsto{\mathsf{p}_2} (\beta, \gamma)$ respectively, and natural transformations $\imath\colon \mathsf{p}_0 \Rightarrow \mathsf{p}_1$ and $\jmath\colon \mathsf{p}_1 \Rightarrow \mathsf{p}_2$ whose components are given by $(\alpha, \beta) \xmapsto{\imath} (\alpha, \gamma)$ and $(\alpha, \gamma) \xmapsto{\jmath} (\beta, \gamma)$ respectively.

DEFINITION 4.1 (cf. [63]). Let $(\mathsf{P}, \leqslant)$ be a finite poset. A *Cartan-Eilenberg system* over $\mathsf{O}(\mathsf{P})$ consists of a covariant functor $\mathsf{E}\colon \mathsf{O}(\mathsf{P})^{\mathbf{2}} \to R\text{-}\mathbf{Mod}$[1] and a natural transformation $k\colon \mathsf{E}\mathsf{p}_2 \Rightarrow \mathsf{E}\mathsf{p}_0$ between the composite functors $\mathsf{E}\mathsf{p}_2$ and $\mathsf{E}\mathsf{p}_0$, called the *connecting homomorphism*, such that

$$\begin{tikzcd} \mathsf{E}\mathsf{p}_0 \arrow[rr, Rightarrow, "\mathsf{E}\imath"] & & \mathsf{E}\mathsf{p}_1 \arrow[dl, Rightarrow, "\mathsf{E}\jmath"] \\ & \mathsf{E}\mathsf{p}_2 \arrow[ul, Rightarrow, "k"] & \end{tikzcd}$$

is an exact triangle, where the natural transformations $\mathsf{E}\imath$ and $\mathsf{E}\jmath$ are the right whiskerings of E and $\imath$, and E and $\jmath$ respectively. A Cartan-Eilenberg system over $\mathsf{O}(\mathsf{P})$ is denoted by $\mathbf{E} = \big(\mathsf{O}(\mathsf{P})^{\mathbf{2}}, \mathsf{E}, k\big)$.[2]

Unpacking the above definition yields

$$\mathsf{E}\colon \mathsf{O}(\mathsf{P})^{\mathbf{2}} \longrightarrow R\text{-}\mathbf{Mod}, \quad (\alpha, \beta) \mapsto \mathsf{E}(\alpha, \beta) = E_\alpha^\beta \in R\text{-}\mathbf{Mod}.$$

The functor E yields the homomorphisms $\ell\colon E_\alpha^\beta \to E_\gamma^\delta$ for all $(\alpha, \beta) \leqslant (\gamma, \delta)$, and the composition $E_\alpha^\beta \xrightarrow{\ell} E_\gamma^\delta \xrightarrow{\ell} E_\epsilon^\zeta$ is given by $\ell\colon E_\alpha^\beta \to E_\epsilon^\zeta$ by the transitivity in $\mathsf{O}(\mathsf{P})$. The natural transformation k yields the differential $k\colon E_\beta^\gamma \to E_\alpha^\beta$ such that the diagrams[3]

$$\begin{tikzcd} E_\alpha^\beta \arrow[rr, "i"] & & E_\alpha^\gamma \arrow[dl, "j"] \\ & E_\beta^\gamma \arrow[ul, "k"] & \end{tikzcd} \qquad \begin{tikzcd} E_\beta^\gamma \arrow[r, "k"] \arrow[d, "\ell"'] & E_\alpha^\beta \arrow[d, "\ell"] \\ E_\epsilon^\zeta \arrow[r, "k"] & E_\delta^\epsilon \end{tikzcd} \tag{4.1}$$

are exact and commutative for all $(\alpha, \beta, \gamma) \leqslant (\delta, \epsilon, \zeta)$. By construction $\ell\colon E_\alpha^\beta \xrightarrow[\mathrm{id}]{\ell} E_\alpha^\beta$ is the identity homomorphism and the exactness of (4.1)[left] shows that $E_\alpha^\alpha = 0$ for all $\alpha \in \mathsf{O}(\mathsf{P})$, cf. [35]. Morphisms between Cartan-Eilenberg systems $\mathbf{E}$ and $\mathbf{E}'$

[1]The category of R-modules is denoted by R-**Mod**. Cartain-Eilenberg systems can be formulated in any abelian category such as abelian groups, R-modules or $\mathbb{K}$-vector spaces.

[2]Cartan-Eilenberg systems can be defined over any poset, eg. all subsets of a topological space, closed subsets, etc., cf. [63].

[3]In the special cases $(\alpha, \beta) \leqslant (\alpha, \gamma)$ and $(\alpha, \gamma) \leqslant (\beta, \gamma)$ the morphisms ℓ are denoted by i and j respectively.

can be described in terms of the E-terms, i.e. a morphism is a natural transformation $h\colon \mathsf{E} \Rightarrow \mathsf{E}'$ which, in terms of E-terms, implies that there exist homomorphisms $h_\alpha^\beta\colon E_\alpha^\beta \to E_\alpha'^\beta$ which commute with the morphisms in **E** and **E**$'$ respectively:

$$
\begin{tikzcd}
{} \arrow[r] & E_\alpha^\beta \arrow[r, "i"] \arrow[d, "h_\alpha^\beta"'] & E_\alpha^\gamma \arrow[r, "j"] \arrow[d, "h_\alpha^\gamma"'] & E_\beta^\gamma \arrow[r, "k"] \arrow[d, "h_\beta^\gamma"'] & E_\alpha^\beta \arrow[r] \arrow[d, "h_\alpha^\beta"'] & {} \\
{} \arrow[r] & E_\alpha'^\beta \arrow[r, "i'"] & E_\alpha'^\gamma \arrow[r, "j'"] & E_\beta'^\gamma \arrow[r, "k'"] & E_\alpha'^\beta \arrow[r] & {}
\end{tikzcd}
\tag{4.2}
$$

for every ordered triple(α, β, γ).

Since $\mathsf{O}(\mathsf{P})$ is a finite lattice it suffices to define a Cartan-Eilenberg system with exact triangle and commutative squares for ordered pairs called an *exact couple system*, cf. [48]. To be more specific we consider the diagrams:

$$
\begin{tikzcd}
E_\varnothing^\alpha \arrow[rr, "i"] & & E_\varnothing^\beta \arrow[dl, "j"] \\
& E_\alpha^\beta \arrow[ul, "k"] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
E_\alpha^\beta \arrow[r, "k"] \arrow[d, "\ell"'] & E_\varnothing^\alpha \arrow[d, "i"] \\
E_\gamma^\delta \arrow[r, "k"] & E_\varnothing^\gamma
\end{tikzcd}
\tag{4.3}
$$

which are exact and commutative for all $(\alpha, \beta) \leqslant (\gamma, \delta)$. For an ordered triple (α, β, γ) the composition $E_\beta^\gamma \xrightarrow{k} E_\varnothing^\beta \xrightarrow{j} E_\alpha^\beta$ defines the connecting homomorphism (differential) $k_{\alpha\beta\gamma}\colon E_\beta^\gamma \to E_\alpha^\beta$. Since (4.3) is exact for $(\beta, \gamma) \in \mathsf{O}(\mathsf{P})^{\mathbf{2}}$ we have that $kj = 0$ and thus $k_{\alpha\beta\gamma}k_{\beta\gamma\delta} = j(kj)k = 0$. Any ordered triple (α, β, γ) yields the following octahedral diagram:[4]

$$
\begin{tikzcd}
& & E_\varnothing^\alpha \arrow[ddll, bend right, "i"'] \arrow[ddrr, bend left, "i"] & & \\
& E_\alpha^\beta \arrow[ur, "k"] \arrow[rr, dashed, "l"'] & & E_\alpha^\gamma \arrow[ul, "k"'] \arrow[dl, dashed, "l"] & \\
E_\varnothing^\beta \arrow[ur, "j"] \arrow[rrrr, bend right, "i"'] & & E_\beta^\gamma \arrow[ll, "k"] \arrow[ul, dashed, "\tilde{k}"] & & E_\varnothing^\gamma \arrow[ul, "j"'] \arrow[ll, "j"]
\end{tikzcd}
\tag{4.4}
$$

where the inner exact triangle (dashed) is induced by the three outer exact triangles, cf. [48, Lem. 4.8].

THEOREM 4.2 (cf. [48], Lem. 4.8). *An exact couple system over* $\mathsf{O}(\mathsf{P})$ *extends to a Cartan-Eilenberg system over* $\mathsf{O}(\mathsf{P})$.

---

[4]If there is no ambiguity about the domain and codomain the sub-indices are omitted from the maps k, i, j, ℓ and $\tilde{k}$.

4.1.2. The excisive property. For most algebraic topological applications of Cartan-Eilenberg systems the excisive property of homology plays an important role which leads to the following definition.

DEFINITION 4.3. A Cartan-Eilenberg system $\mathbf{E}$ over $\mathsf{O}(\mathsf{P})$ is called *excisive* if

$$E^{\beta}_{\alpha\cap\beta} \cong E^{\alpha\cup\beta}_{\alpha}, \quad \forall \alpha,\beta \in \mathsf{O}(\mathsf{P}).^{5} \tag{4.5}$$

Two ordered pairs $(\alpha,\beta),(\alpha',\beta') \in \mathsf{O}(\mathsf{P})^{\mathbf{2}}$ are *equivalent*, if $\beta \smallsetminus \alpha = \beta' \smallsetminus \alpha'$. The excisive property for a Cartan-Eilenberg system implies that the E-terms only depend on equivalent pairs up to isomorphism. For $\beta\smallsetminus\alpha = \{p\}$ we abuse notation and write $E_p := E^{\beta}_{\alpha}$ for all $p \in \mathsf{P}$.

LEMMA 4.4. *Let (α,β) and (α',β') be equivalent pairs in $\mathsf{O}(\mathsf{P})^{\mathbf{2}}$. Then, $E^{\beta}_{\alpha} \cong E^{\beta'}_{\alpha'}$.*

PROOF. Define $\tilde\alpha = \alpha \vee \alpha'$ and $\tilde\beta = \beta \vee \beta'$. Then, $\alpha,\alpha' \leqslant \tilde\alpha$, $\beta,\beta' \leqslant \tilde\beta$ and $\beta \smallsetminus \alpha = \beta' \smallsetminus \alpha' = \tilde\beta \smallsetminus \tilde\alpha$. Consider $\alpha \leqslant \tilde\alpha$ and $\beta \leqslant \tilde\beta$. Then, $(\beta \cap \tilde\beta) \smallsetminus (\alpha \cup \tilde\alpha) = \beta \smallsetminus \tilde\alpha = \tilde\beta\smallsetminus\tilde\alpha$. Consequently, $(\beta\cup\tilde\alpha)\smallsetminus\tilde\alpha = \tilde\beta\smallsetminus\tilde\alpha$ and therfore $\tilde\beta = \beta\cup\tilde\alpha$. Similarly, $\beta \smallsetminus (\beta\cap\tilde\alpha) = \beta\smallsetminus\alpha$ which implies that $\alpha = \beta\wedge\tilde\alpha$. By (4.5) we conclude that $E^{\beta}_{\alpha} = E^{\beta}_{\tilde\alpha\cap\beta} \cong E^{\tilde\alpha\cup\beta}_{\tilde\alpha} = E^{\tilde\beta}_{\tilde\alpha}$, By the same token one proves that $E^{\beta'}_{\alpha'} = E^{\beta'}_{\tilde\alpha\cap\beta'} \cong E^{\tilde\alpha\cup\beta'}_{\tilde\alpha} = E^{\tilde\beta}_{\tilde\alpha}$, and thus $E^{\beta}_{\alpha} \cong E^{\beta'}_{\alpha'}$. □

Excisive Cartan-Eilenberg systems already appear in the seminal work by Franzosa on connection matrices for Morse representations, cf. [21]. In Franzosa's work such data structures of R-modules of $\mathbb{K}$-vector spaces are referred to a *module braids*. As a matter of fact one can prove that these concepts are equivalent.

THEOREM 4.5 (cf. [63]). *A module braid over the convex sets in P is equivalent to an excisive Cartan-Eilenberg system over $\mathsf{O}(\mathsf{P})$.*

An important result for excisive Cartan-Eilenberg systems is a representation in terms of finitely generated differential modules. For convenience we assume that the ring R is a *principal ideal domain*. Recall that P-graded differential module is denoted by (C,d), with $C = \bigoplus_{p\in\mathsf{P}} G_pC$, cf. App. C.2-C.3. A P-graded differential module (C,d) is free if and only if the components G_pC are free. Free graded differential modules are used to construct representations of Cartan-Eilenberg systems. A P-graded differential module defines an $\mathsf{O}(\mathsf{P})$-filtered differential module via $\alpha \mapsto F_\alpha C := \bigoplus_{p\in\alpha} G_pC$, cf. App. C.2-C.3. In general, $\mathsf{O}(\mathsf{P})$-filtered differential modules induce excisive Cartan-Eilenberg systems. Consider the short exact sequence:

$$0 \longrightarrow F_\alpha C \xrightarrow[\subset]{i} F_\beta C \xrightarrow{j} \frac{F_\beta C}{F_\alpha C} \longrightarrow 0\ , \quad \alpha \leqslant \beta. \tag{4.6}$$

Since the differential d preserves the filtering we may define the homologies $E^{\alpha}_{\varnothing} := H(F_\alpha C,\mathrm{d})$, $E^{\beta}_{\varnothing} := H(F_\beta C,\mathrm{d})$ and $E^{\beta}_{\alpha} := H\big(F_\beta C/F_\alpha C,\mathrm{d}\big)$. This yields the exact

[5]The homomorphism $\ell\colon E^{\beta}_{\alpha\cap\beta} \to E^{\alpha\cup\beta}_{\alpha}$ is an isomorphism.

triangles in (4.3) where $k\colon H\big(F_\beta C/F_\alpha C,\mathrm{d}\big)\to H(F_\alpha C,\mathrm{d})$ is the connecting homomorphism constructed in the usual way. All other axioms of exact couple systems are readily verified which yields a Cartan-Eilenberg system denoted $\mathbf{E}(C,\mathrm{d})$. The excisive property follows from the fact that $F_{\alpha\cup\beta}C/F_\alpha C = (F_\alpha C + F_\beta C)/F_\alpha C \cong F_\beta C/(F_\alpha C\cap F_\beta C) = F_\beta/F_{\alpha\cap\beta}C$. The (excisive) Cartan-Eilenberg system $\mathbf{E}(C,\mathrm{d})$ is the Cartan-Eilenberg system of the $\mathsf{O}(\mathsf{P})$-filtered differential module (C,d). This implies that a Cartan-Eilenberg system of a P-graded differential module is automatically defined.

DEFINITION 4.6. Let $\mathbf{E}$ be a Cartan-Eilenberg system over a finite distributive lattice $\mathsf{O}(\mathsf{P})$. A free, P-graded differential group (C,d), with $C=\bigoplus_{p\in\mathsf{P}}G_pC$, is a P-*graded representation* for $\mathbf{E}$ if

$$\mathbf{E}(C,\mathrm{d})\cong\mathbf{E}. \tag{4.7}$$

A P-graded representation is *strict* if the free P-graded differential group (C,d) is strict.[6]

The main theorem of this section states that in most cases P-graded representations for Cartan-Eilenberg systems exist and are unique up to conjugacy. The existence part was proved in [21] and applies to Cartan-Eilenberg systems due to Theorem 4.5. The existence result in [21] assumes that every E-term is the homology of a differential module. We say in this case that the Cartan-Eilenberg system is *chain generated*. To be more precise, for all $(\alpha,\beta)\in\mathsf{O}(\mathsf{P})^{\mathbf{2}}$ there exist differential modules $\big(C^\alpha_\varnothing,d^\alpha_\varnothing\big)$, $\big(C^\beta_\varnothing,d^\beta_\varnothing\big)$ and $\big(C^\beta_\alpha,d^\beta_\alpha\big)$, and short exact sequences

$$0\longrightarrow C^\alpha_\varnothing\xrightarrow{\ i\ }C^\beta_\varnothing\xrightarrow{\ j\ }C^\beta_\alpha\longrightarrow 0,$$

such that $E^\alpha_\varnothing = H\big(C^\alpha_\varnothing,d^\alpha_\varnothing\big)$, $E^\beta_\varnothing = H\big(C^\beta_\varnothing,d^\beta_\varnothing\big)$ and $E^\beta_\alpha = H\big(C^\beta_\alpha,d^\beta_\alpha\big)$. By the standard construction of the connecting homomorphisms this yields a Cartan-Eilenberg system $\mathbf{E}$. This concept is more general than a Cartan-Eilenberg system generated by an $\mathsf{O}(\mathsf{P})$-filtered differential module, or a P-graded differential module, cf. [63].

THEOREM 4.7 ([21], Thm. 4.8). *Let* $\mathbf{E}$ *be a chain generated, excisive Cartan-Eilenberg system over* $\mathsf{O}(\mathsf{P})$. *Then, there exists a free,* P*-graded differential group* (C,d) *— a* P*-graded representation — such that* $\mathbf{E}\cong\mathbf{E}(C,\mathrm{d})$.

An R-module is *finitely generated* if it has a finite generating set. The running assumption in this chapter is that R is a principal ideal domain. This implies that a module $C\cong R^n\oplus\mathrm{Tor}(C)$, where $\mathrm{Tor}(C)$ is the *maximal torsion submodule* of C[7] and $\mathrm{Tor}(C)\cong\bigoplus_i R/(d_i)$, where d_i are the non-zero invariant factors of C. The integer n is called the *rank* of C. A Cartan-Eilenberg system $\mathbf{E}$ is finitely generated if all

[6]Recall that (C,d) is strict if the differential restricted to G_pC, $p\in\mathsf{P}$, is trivial, cf. Defn. C.3.

[7]Recall that $c\in\mathrm{Tor}(C)$ if the exists an $r\in R$ such that $rc=0$.

modules E_α^β, $(\alpha,\beta) \in \mathsf{O}(\mathsf{P})^{\mathbf{2}}$. Suppose $\mathbf{E}$ is a finitely generated, excisive Cartan-Eilenberg system over $\mathsf{O}(\mathsf{P})$. For E_α^β let s_α^β be the rank of E_α^β and r_α^β is the number of non-zero invariant factors of the maximal torsion submodule $\mathrm{Tor}(E_\alpha^\beta)$. For $\beta \smallsetminus \alpha = \{p\}$ we denote s_α^β and r_α^β by s_p and r_p respectively.

DEFINITION 4.8. Let $\mathbf{E}$ be a finitely generated, excisive Cartan-Eilenberg system over a finite distributive lattice $\mathsf{O}(\mathsf{P})$. A free, P-graded differential module (C, d), with $C = \bigoplus_{p\in\mathsf{P}} G_pC$, is a *principal representation* for $\mathbf{E}$ if

(i) $\mathbf{E}(C,\mathrm{d}) \cong \mathbf{E}$;
(ii) $\operatorname{rank} G_pC = s_p + 2r_p$ for all $p \in \mathsf{P}$.

The differential d is called a *spectral matrix* for $\mathbf{E}$.

Since R-modules over a principal ideal domain allow length 1 free resolutions the existence of a principal representation is guaranteed by Theorem 4.7. The main result in [63] states that such a representation is unique up to isomorphism of Cartan-Eilenberg systems which implies that differentials (spectral matrices) are unique up to conjugacy.

REMARK 4.9. For strict P-graded differential modules the homology satisfies $H(G_pC,\mathrm{d}) = G_pC \cong E_\alpha^\beta$, with $\beta \smallsetminus \alpha = \{p\}$ for all $p \in \mathsf{P}$.

REMARK 4.10. In the context of dynamical systems a spectral matrix is referred to as a *connection matrix*. We will refer to this nomenclature when we apply Cartan-Eilenberg systems for bi-topological spaces involving the block-flow topology.

REMARK 4.11. If a Cartan-Eilenberg system is generated by an $\mathsf{O}(\mathsf{P})$-filtered differential $\mathbb{K}$-vector space, then [58] provides a simplified proof of Theorem 4.7.

4.1.3. Cartan-Eilenberg systems of a filtered topological space. As before let $(\mathsf{P}, \leqslant)$ be a finite poset and $(X, \mathscr{T})$ be a topological space. Consider a P-graded decomposition of X given by $X = \bigcup_{p\in\mathsf{P}} G_pX$, cf. App. C.1. Dual to a grading is a lattice filtering $\mathrm{grd}^{-1}\colon \mathsf{O}(\mathsf{P}) \to \mathsf{Set}(X)$ given by the lattice homomorphism $\alpha \mapsto F_\alpha X := \mathrm{grd}^{-1}\alpha$.[8] From this point on one can invoke a (generalized) (co)homology theory by assigning $E_\alpha^\beta := H(F_\beta X, F_\alpha X)$ for every $(\alpha,\beta) \in \mathsf{O}(\mathsf{P})^{\mathbf{2}}$. From the Eilenberg-Steenrod axioms we have exact triangles (and the connecting homomorphisms) and commutative squares:

$$
\begin{tikzcd}
H(F_\alpha X) \arrow[rr, "i"] & & H(F_\beta X) \arrow[dl, "j"] \\
& H(F_\beta X, F_\alpha X) \arrow[ul, "k"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
H(F_\beta X, F_\alpha X) \arrow[r, "k"] \arrow[d, "\ell"] & H(F_\alpha X) \arrow[d, "i"] \\
H(F_\delta X, F_\gamma X) \arrow[r, "k"] & H(F_\gamma X)
\end{tikzcd}
$$

[8] If $(\mathsf{P}, \leqslant)$ is $\mathscr{T}$-consistent then tile is a continuous map, and $\mathrm{grd}^{-1}\colon \mathsf{O}(\mathsf{P}) \to \mathscr{C}(X,\mathscr{T})$ is filtering of closed set of X. The mapping $\mathrm{grd}^{-1}\colon \mathsf{U}(\mathsf{P}) \to \mathscr{O}(X,\mathscr{T})$ yields a filtering of open sets in X. This can also be obtained by considering a $\mathscr{T}$-co-consistent grading, i.e. continuous with respect to opposite poset P^*.

for $(\alpha,\beta) \leqslant (\gamma,\delta)$. For example the homology functor can be taken to be singular homology. Note that we suppress the $\mathbb{Z}$-grading as the Cartan-Eilenberg theory works in the setting of differential modules. Singular homology yields a chain generated Cartan-Eilenberg system $\mathbf{E}^{\text{sing}}(X)$. Denote by $C(X)$ the R-module of singular chains over a ring R and $\mathrm{d}\colon C(X) \to C(X)$ is the singular boundary operator, or differential making $\big(C(X),\mathrm{d}\big)$ a differential module.[9] The filtering of X yields a filtering of $C(X)$, i.e. $\alpha \mapsto F_\alpha C(X)$ with $F_\alpha C(X) := C\big(F_\alpha X\big)$, where $C\big(F_\alpha X\big)$ are the singular chains in $C(X)$ restricted to $F_\alpha X$. For singular chains it holds that $F_{\alpha\cap\beta}C(X) = F_\alpha C(X) \cap F_\beta C(X)$. The same relation with respect to union does not hold in general. The differential satisfies $\mathrm{d}F_\alpha C(X) \subset F_\alpha C(X)$ for all $\alpha \in \mathsf{O}(\mathsf{P})$, making $\alpha \mapsto F_\alpha C(X)$ a meet semi-lattice filtered module (chain complex). The fact that the latter is not an $\mathsf{O}(\mathsf{P})$-filtered module prevents us from regarding $\mathbf{E}^{\text{sing}}(X)$ as generated by an $\mathsf{O}(\mathsf{P})$-filtered differential module. However, for the filtering we obtain the following short exact sequences:

$$0 \longrightarrow C(F_\alpha X) \xrightarrow{\;i\;} C(F_\beta X) \xrightarrow{\;j\;} \frac{C(F_\beta X)}{C(F_\alpha X)} \longrightarrow 0,$$

$(\alpha,\beta) \in \mathsf{O}(\mathsf{P})^{\mathbf{2}}$, which represent the modules $C^\alpha_\varnothing$, $C^\beta_\varnothing$ and C^β_α. The associated homologies $H(F_\alpha X) = H\big(C(F_\alpha X),\mathrm{d}\big)$, $H(F_\beta X) = H\big(C(F_\beta X),\mathrm{d}\big)$ and

$$H(F_\beta X, F_\alpha X) = H\big(C(F_\beta X)/C(F_\alpha X),\mathrm{d}\big),$$

yield the above exact triangle for a pair. We conclude that $\mathbf{E}^{\text{sing}}(X)$ is chain generated as explained in Section 4.1.2. The fact that $(C(X),\mathrm{d})$ is not lattice filtered implies that the associated Cartan-Eilenberg system is *not* excisive in general. Depending on the homology theory, or on properties of the grading of X, we can relate the homologies $H(F_\beta X, F_\alpha X)$ and $H\big(F_\beta X/F_\alpha X\big)$ which are not necessarily isomorphic. If a grading $X = \bigcup_{p\in\mathsf{P}} G_p X$ is *natural* then the associated filtering $\alpha \mapsto F_\alpha X$ consists of mutually good pairs, cf. Defn. 2.17. For example if we consider singular homology then $H(F_\beta X, F_\alpha X) \cong H\big(F_\beta X/F_\alpha X\big)$ for all $\alpha \subset \beta$, cf. [34, Prop. 2.22], [69, Thm. 3.2.9]. In this case the relative singular homology satisfies the excisive property, i.e. $H(F_{\alpha\cup\beta}X, F_\alpha X) \cong H\big(F_{\alpha\cup\beta}X/F_\alpha X\big) \cong H\big(F_\beta X/F_{\alpha\cap\beta}X\big) \cong H(F_\beta X, F_{\alpha\cap\beta}X)$, for all $\alpha,\beta \in \mathsf{O}(\mathsf{P})$. If X is a compact Hausdorff space and the poset P is $\mathscr{T}$-consistent (not necessarily natural), i.e. the filtering consists of closed sets $F_\alpha X \in \mathscr{C}(X,\mathscr{T})$. By the closedness of $F_\alpha X$ we have the homeomorphisms $F_\beta X/F_\alpha X \smallsetminus [F_\alpha X] \cong F_\beta X \smallsetminus F_\alpha X$, for all $\alpha \subset \beta$. Let $\bar{H}$ represent *Alexander-Spanier cohomology*. From the excisive property of Alexander-Spanier cohomology we have that[10]

$$\begin{aligned}\bar{H}\big(F_\beta X, F_\alpha X\big) &\cong \bar{H}_c\big(F_\beta X \smallsetminus F_\alpha X\big) \cong \bar{H}_c\big(F_\beta X/F_\alpha X \smallsetminus [F_\alpha X]\big)\\ &\cong \bar{H}\big(F_\beta X/F_\alpha X, [F_\alpha X]\big) =: \bar{H}\big(F_\beta X/F_\alpha X\big),\end{aligned}$$

[9]If we invoke $C(X) = \bigoplus_q C_q(X)$ as the $\mathbb{Z}$-graded module of singular chains then (C,d) is a chain complex.

[10]Here $\bar{H}_c$ denote compactly supported Alexander-Spanier cohomology.

cf. [62, Ch. 6, Sect. 6, Lem. 11], [16, Ch. V, Sect. 2.A]. The same result can be obtained by using compactly supported *Alexander-Spanier homology*, cf. [47, Cor. 9.4].

REMARK 4.12. Suppose X is a (locally) compact Hausdorff space homeomorphic to a finite CW-complex and the poset P is $\mathscr{T}$-consistent such that $F_\alpha X$ is homeomorphic to a closed subcomplex. Then, $F_\beta X/F_\alpha X$ is the one-point compactification of $G_{\beta\smallsetminus\alpha}X := F_\beta X \smallsetminus F_\alpha X$. For the singular homology we have the isomorphism $H\big(F_\beta X/F_\alpha X\big) \cong H^{\mathrm{BM}}\big(G_{\beta\smallsetminus\alpha}X\big)$, where H^{BM} indicates the *Borel-Moore homology* of $G_{\beta\smallsetminus\alpha}X$, cf. [8]. As a matter of fact the Borel-Moore chain groups $C^{\mathrm{BM}}\big(F_\beta X \smallsetminus F_\alpha X\big)$ yield a short exact sequences

$$0 \to C^{\mathrm{BM}}\big(F_\beta X \smallsetminus F_\alpha X\big) \xrightarrow{i} C^{\mathrm{BM}}\big(F_\gamma X \smallsetminus F_\alpha X\big) \xrightarrow{j} C^{\mathrm{BM}}\big(F_\gamma X \smallsetminus F_\beta X\big) \to 0, \tag{4.8}$$

as opposed to the weakly exact sequences in (4.9).

REMARK 4.13. In [21, 19] considers the sequence of pointed quotient spaces $F_\beta X/F_\alpha X$ which induces the weakly exact sequence[11]

$$C(F_\alpha X) \xrightarrow{\ \ i\ \ } C(F_\beta X) \xrightarrow{\ \ j\ \ } C\big(F_\beta X/F_\alpha X\big) \tag{4.9}$$

of singular chains on the quotient spaces for good pairs. The approach in [19] allows slightly weaker conditions on the good pairs, cf. [46].

4.2. Tessellar homology

The objective of the homological algebra in this section is to obtain algebraic topological invariants of X via finite algebraic information; *discretization of algebraic topology*. To do so we employ the abstract formalism of Cartan-Eilenberg systems as explained in Section 4.1. Let $\mathrm{disc}\colon (X,\mathscr{T}) \twoheadrightarrow (\mathfrak{X},\leqslant)$ be a natural discretization map, i.e. $\leqslant$ is a $\mathscr{T}$-consistent pre-order and consist of mutually good pairs. Let $\mathfrak{X}/_\sim$ be the poset of partial equivalence classes of $(\mathfrak{X},\leqslant)$. Then, the map $X \twoheadrightarrow \mathfrak{X}/_\sim$ given by the composition

$$X \xrightarrow{\ \ \mathrm{disc}\ \ } \mathfrak{X} \xrightarrow{\ \ \pi\ \ } \mathfrak{X}/_\sim, \tag{4.10}$$

is natural and yields a $\mathfrak{X}/_\sim$-grading $X = \bigcup_{[\xi]} G_{[\xi]}X$, cf. Rem. 2.10 and App. C.1. The associated filtering $\mathcal{U} \mapsto F_{\mathcal{U}}X$, $\mathcal{U} \in \mathsf{O}(\mathfrak{X},\leqslant)$, defined by disc^{-1}, consists of good pairs and yields an excisive Cartan-Eilenberg system $\mathbf{E}^{\mathrm{disc}}$ as outlined in Section 4.1.3. For simplicity we assume that $\mathbf{E}^{\mathrm{disc}}$ is *finitely generated* for the remainder of this chapter. We now explain the construction of an associated homology theory.

---

[11] Weakly exact sequences yield exact triangles homology. Recall that a seqeunce $A \xrightarrow{i} B \xrightarrow{j} C$ is *weakly exact* is i is injective, $j \circ i = 0$ and the quotient map $B/\mathrm{im}\, i \to C$ induced by j yields an isomorphism $H(B/\mathrm{im}\, j) \cong H(C)$, cf. [46] and [21] for more detail.

4.2.1. The tessellar differential module. For $\xi \in \mathfrak{X}$ define the E-terms for $\mathbf{E}^{\mathrm{disc}}$ via relative (singular) homology $E_{[\xi]} := H\big(F_{\downarrow\xi}X, F_{\downarrow\xi^{\blacktriangleleft}}X\big)$, which is chain generated and finitely generated with *coefficients in a principal ideal domain* R. The *tessellar modules*, or *tessellar chain groups* are given by the external direct sum

$$C^{\mathrm{disc}}(X) := \bigoplus_{[\xi]\in\mathfrak{X}/_{\sim}} G_{[\xi]}C^{\mathrm{disc}}(X), \tag{4.11}$$

where $G_{[\xi]}C^{\mathrm{disc}}(X) := H\big(F_{\downarrow\xi}X, F_{\downarrow\xi^{\blacktriangleleft}}X\big)$[12] if the latter is a free R-module, or else choose a free differential module $\big(G_{[\xi]}C^{\mathrm{disc}}(X), \mathrm{d}\big)$ such that

(i) $H\big(G_{[\xi]}C^{\mathrm{disc}}, \mathrm{d}\big) \cong H\big(F_{\downarrow\xi}X, F_{\downarrow\xi^{\blacktriangleleft}}X\big)$;
(ii) $G_{[\xi]}C^{\mathrm{disc}}(X) \cong R^{s_\xi+2r_\xi}$, cf. Defn. 4.8(ii).

By Theorem 4.7 there exists an $\mathsf{O}(\mathfrak{X})$-filtered differential

$$\mathrm{d}^{\mathrm{disc}}\colon C^{\mathrm{disc}}(X) \to C^{\mathrm{disc}}(X)$$

such that $\mathbf{E}\big(C^{\mathrm{disc}}, \mathrm{d}^{\mathrm{disc}}\big) \cong \mathbf{E}^{\mathrm{disc}}$ and the spectral matrix $\mathrm{d}^{\mathrm{disc}}$ is unique up to isomorphism, cf. Sect. 4.1.2. If the homologies $H\big(F_{\downarrow\xi}X, F_{\downarrow\xi^{\blacktriangleleft}}X\big)$ are free (finitely generated), then $\big(C^{\mathrm{disc}}, \mathrm{d}^{\mathrm{disc}}\big)$ is a strict $\mathfrak{X}/_{\sim}$-graded differential module with $C^{\mathrm{disc}}(X) = \bigoplus_{[\xi]\in\mathfrak{X}/_{\sim}} H\big(F_{\downarrow\xi}X, F_{\downarrow\xi^{\blacktriangleleft}}X\big)$. We refer to $\big(C^{\mathrm{disc}}, \mathrm{d}^{\mathrm{disc}}\big)$ as the $\mathfrak{X}/_{\sim}$-graded *tessellar differential module*, or $\mathfrak{X}/_{\sim}$-graded *tessellar chain complex* based on singular homology. The associated homology is called the *tessellar homology* of $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ and is denoted by $H^{\mathrm{disc}}(X) \cong H(X)$. Following Appendix C.2 the restricted module $G_{\mathcal{U}\smallsetminus\mathcal{U}'}C^{\mathrm{disc}}(X)$ is well-defined for every convex set $\mathcal{U}\smallsetminus\mathcal{U}'$, with $\mathcal{U},\mathcal{U}' \in \mathsf{O}(\mathfrak{X},\leqslant)$ and

$$G_{\mathcal{U}\smallsetminus\mathcal{U}'}C^{\mathrm{disc}}(X) := \bigoplus_{[\xi]\subset\mathcal{U}\smallsetminus\mathcal{U}'} G_{[\xi]}C^{\mathrm{disc}} = \frac{F_{\mathcal{U}}C^{\mathrm{disc}}}{F_{\mathcal{U}'}C^{\mathrm{disc}}}. \tag{4.12}$$

The differential is the restriction of $\mathrm{d}^{\mathrm{tess}}$ to $G_{\mathcal{U}\smallsetminus\mathcal{U}'}C^{\mathrm{tess}}(X)$. The associated homology $H^{\mathrm{disc}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'}X) := H\big(G_{\mathcal{U}\smallsetminus\mathcal{U}'}C^{\mathrm{disc}}(X), \mathrm{d}^{\mathrm{disc}}\big)$ is the tessellar homology of the locally closed set[13] $G_{\mathcal{U}\smallsetminus\mathcal{U}'}X$. The latter also defines the relative tessellar homology $H^{\mathrm{disc}}\big(F_{\mathcal{U}}X, F_{\mathcal{U}'}X\big)$.

THEOREM 4.14. *Let* $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ *be a natural discretization map. Then, the tessellar homology satisfies* $H^{\mathrm{disc}}(X) \cong H(X)$. *In particular, for every convex set* $\mathcal{U}\smallsetminus\mathcal{U}' \subset \mathfrak{X}$, $\mathcal{U},\mathcal{U}' \in \mathsf{O}(\mathfrak{X},\leqslant)$, *we have that* $H^{\mathrm{disc}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'}X) \cong H(F_{\mathcal{U}}X, F_{\mathcal{U}'}X)$.

PROOF. By definition the modules $G_{[\xi]}C^{\mathrm{disc}}$ are free. By [21, Thm. 4.8] we have a differential $\mathrm{d}^{\mathrm{disc}}$ which retrieves the homologies. □

The discretization map $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ discretizes X, while the construction of $\big(C^{\mathrm{disc}}(X), \mathrm{d}^{\mathrm{disc}}\big)$ discretizes the algebraic topology of X. More details on tessellar homology are discussed in [63].

12By the isomorphism $\mathsf{O}(\mathfrak{X}/_{\sim}) \cong \mathsf{O}(\mathfrak{X},\leqslant)$ we have that $\downarrow\xi = \downarrow[\xi]$.

13A locally closed subset of X is given as the intersection an open and a closed subset in X.

4.2.2. The tessellar complex. It it sometimes useful to describe the tessellar homology in combinatorial terms. To do some we define a slight generalization of a (Lefschetz) complex, cf. [31].

DEFINITION 4.15. A *complex* is a triple $(\mathcal{T}, \leqslant, \kappa)$ where $(\mathcal{T}, \leqslant)$ is a finite pre-order and $\kappa\colon \mathcal{T}\times\mathcal{T}\to R$ is a function satisfying

(i) (upper-triangular) $\kappa(\vartheta,\vartheta')\neq 0$ implies $\vartheta\leqslant\vartheta'$, $\vartheta\neq\vartheta'$;
(ii) (boundary) $\sum_{\vartheta'}\kappa(\vartheta,\vartheta')\kappa(\vartheta',\vartheta'')=0$.

The function κ is called the *incidence function* and its values in the ring R are called the *incidence numbers*.

For a cell complex we can define (free) canonical differential R-module. Define the free R-module over $\mathcal{T}$:

$$C(\mathcal{T}) := \bigoplus_{\vartheta\in\mathcal{T}} R\langle\vartheta\rangle^{14}$$

with differential

$$\mathrm{d}_{\mathcal{T}}\langle\vartheta\rangle := \sum_{\vartheta'\in\mathcal{T}} \kappa(\vartheta,\vartheta')\langle\vartheta'\rangle,$$

which makes $\big(C(\mathcal{T}),\mathrm{d}_{\mathcal{T}}\big)$ a $\mathcal{T}/_\sim$-graded differential R-module. For every convex set $\mathcal{E}\subset\mathcal{T}$ the restriction of $C(\mathcal{T})$ to $C(\mathcal{E})$ and differential accordingly defines a *subcomplex* $\big(C(\mathcal{E}),\mathrm{d}_{\mathcal{E}}\big)$. The associated Cartan-Eilenberg system is excisive and is denoted by $\mathbf{E}\big(C(\mathcal{T}),\mathrm{d}_{\mathcal{T}}\big)$ with E-terms given by $E_\alpha^\beta := H\big(C(\mathcal{E}),\mathrm{d}_{\mathcal{E}}\big)$, with $\beta\smallsetminus\alpha=\mathcal{E}$. Consider the diagram

$$\begin{tikzcd} X \arrow[r, two heads, "\mathrm{disc}"] & \mathfrak{X} \arrow[r, two heads, "\pi"] & \mathfrak{X}/_\sim \\ & & \mathcal{T} \arrow[u, two heads, "\varpi"'] \end{tikzcd} \tag{4.13}$$

which implies that the equivalence classes in $\mathcal{T}$ coincide with the equivalence classes in $\mathfrak{X}$, i.e. $\mathcal{T}/_\sim\cong\mathfrak{X}/_\sim$.

DEFINITION 4.16. A triple $(\mathcal{T}, \leqslant, \kappa)$ is *tessellar complex* for a natural discretization map $\mathrm{disc}\colon X\twoheadrightarrow\mathfrak{X}$ if if

$$\big(C(\mathcal{T}),\mathrm{d}_{\mathcal{T}}\big)\cong\big(C^{\mathrm{disc}},\mathrm{d}^{\mathrm{disc}}\big),$$

as $\mathsf{O}(\mathfrak{X})$-filtered chain isomorphic $\mathfrak{X}/_\sim$-graded differential modules.

A tessellar complex yields a differential matrix with R-coefficients via the incidence function κ. This justifies the terminology spectral matrix. For a convex set $\mathcal{E}\subset\mathfrak{X}$ we use the notation

$$H^{\mathrm{disc}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'}X)\cong H\big(C(\mathcal{E}),\mathrm{d}_{\mathcal{E}}\big)\cong H(F_{\mathcal{U}}X, F_{\mathcal{U}'}X), \tag{4.14}$$

with $\mathcal{E}=\mathcal{U}\smallsetminus\mathcal{U}'$ is independent of the pair $\mathcal{U}'\subset\mathcal{U}$ and where $\big(C(\mathcal{E}),\mathrm{d}_{\mathcal{E}}\big)$ is restriction of $\big(C(\mathcal{T}),\mathrm{d}_{\mathcal{T}}\big)$ to $\mathcal{E}$. In Section 4.3 we discuss the special case of a CW-decomposition.

[14] We use the notation $\langle\vartheta\rangle$ to express ϑ as basis for $C(\mathcal{T})$.

REMARK 4.17. Given a natural discretization map $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ one can always construct a map $\mathrm{disc}\colon X \to \mathfrak{X}$ (not necessarily surjective) such that there exists a tessellar complex $(\mathfrak{X}, \leqslant, \kappa)$. Such a discretization map is a *representable* natural discretization.

4.2.3. Linear discretization, split grading and bi-graded Betti numbers. In certain cases a natural discretization map $\mathrm{disc}\colon X \twoheadrightarrow \mathfrak{X}$ allows another scalar discretization map via an order-preserving map $\mathrm{ind}\colon \mathfrak{X} \to \mathbb{Z}$ (not necessarily surjective). By construction ind factors through $\mathfrak{X}/_\sim$:

$$\begin{tikzcd} X \arrow[r, two heads, "\mathrm{disc}"] \arrow[rrr, bend right, "\mathrm{skel}"'] & \mathfrak{X} \arrow[r, two heads, "\pi"] \arrow[rr, bend left, "\mathrm{ind}"] & \mathfrak{X}/_\sim \arrow[r] & \mathbb{Z} \end{tikzcd}$$

The map ind maps to a linear order and the composition skel is called a *linear discretization*. The discretization ind is a coarsening of disc and is therefore a natural discretization map, which makes $\big(C^{\mathrm{disc}}, \mathrm{d}^{\mathrm{disc}}\big)$ a $\mathbb{Z}$-graded differential module. Since ind is order preserving the tessellar homology of $G_pX := \mathrm{skel}^{-1}p \subset X$ is well-defined. For the latter we consider a standard spectral sequence. Define, using (4.12),

$$G_pC^{\mathrm{disc}}(X) = \bigoplus_{[\xi]\subset \mathrm{ind}^{-1}p} G_{[\xi]}C^{\mathrm{disc}}(X),$$

which gives the $\mathbb{Z}$-grading $C^{\mathrm{disc}}(X) = \bigoplus_{p\in\mathbb{Z}} G_pC^{\mathrm{tess}}(X)$. As explained in Section 4.1.2 we obtain the short exact sequences

$$0 \longrightarrow F_{\downarrow(p-1)}C^{\mathrm{disc}} \xrightarrow{i_{p-1}} F_{\downarrow p}C^{\mathrm{disc}} \xrightarrow{j_p} G_pC^{\mathrm{disc}} \longrightarrow 0$$

From the tessellar boundary operator $\mathrm{d}^{\mathrm{disc}}$ we compute the homology which provide the zeroth and first pages $\mathscr{E}^0 = \bigoplus_p \mathscr{E}^0_p$ and $\mathscr{E}^1 = \bigoplus_p \mathscr{E}^1_p$ with

$$\mathscr{E}^0_p = G_pC^{\mathrm{disc}}(X) \quad \text{and} \quad \mathscr{E}^1_p = H\big(G_pC^{\mathrm{disc}}\big) \cong H^{\mathrm{disc}}\big(G_pX\big).$$

This yields the exact triangles, using $H^{\mathrm{disc}}(F_{\downarrow p}X) = H\big(F_{\downarrow p}C^{\mathrm{disc}}\big)$,

$$\begin{tikzcd} H^{\mathrm{disc}}(F_{\downarrow(p-1)}X) \arrow[rr, "i_{p-1}"] & & H^{\mathrm{disc}}(F_{\downarrow p}X) \arrow[dl, "j_p"] \arrow[rr, "i_p"] & & H^{\mathrm{disc}}(F_{\downarrow(p+1)}X) \arrow[dl, "j_{p+1}"] \\ & \mathscr{E}^1_p \arrow[ul, "k_p"] & & \mathscr{E}^1_{p+1} \arrow[ul, "k_{p+1}"] \arrow[ll, dotted, bend left, "\mathrm{d}^1_{p+1}"] & \end{tikzcd} \tag{4.15}$$

where $\mathrm{d}^1_p = j_{p-1}k_p$ are the connecting homomorphisms computed from $\mathrm{d}^{\mathrm{disc}}$. Recursively define $\mathscr{E}^{r+1} = H(\mathscr{E}^r, \mathrm{d}^r)$ where $\mathrm{d}^r_p\colon \mathscr{E}^r_p \to \mathscr{E}^r_{p-r}$ with $\mathrm{d}^r = ji^{1-r}k$. Since ind defines a finite filtering on $C^{\mathrm{disc}}(X)$ the spectral sequence converges and

$\mathscr{E}^{\infty}_p \cong G_pH(C^{\mathrm{disc}}) = G_pH^{\mathrm{disc}}(X)$. If we set $\beta^{\mathrm{disc}}_p = \operatorname{rank} G_pH^{\mathrm{disc}}(X)$, called the *Betti numbers*, then

$$\sum_{p\in\mathbb{Z}} \beta^{\mathrm{disc}}_p = \operatorname{rank} H^{\mathrm{disc}}(X).$$

In particular, when $R = \mathbb{K}$, then ind yields a grading on the tessellar homology. For ring coefficients this is more complicated and it is not true in general that $\operatorname{Gr} H^{\mathrm{disc}}(X) = \bigoplus_{p\in\mathsf{P}} G_pH^{\mathrm{disc}}(X)$ is isomorphic to $H^{\mathrm{disc}}(X)$, cf. App. C.2. However, if the differential satisfies $\mathrm{d}^{\mathrm{disc}}G_pC^{\mathrm{disc}} \subset G_{p-1}C^{\mathrm{disc}}$, then the tessellar differential module $\big(C^{\mathrm{disc}}, \mathrm{d}^{\mathrm{disc}}\big)$, with $\mathrm{d}^{\mathrm{disc}}_p := \mathrm{d}^{\mathrm{disc}}(p, p-1)$, [15] is a chain complex $\big(G_pC^{\mathrm{disc}}, \mathrm{d}^{\mathrm{disc}}_p\big)$. We write $\mathrm{d}^{\mathrm{disc}} = \bigoplus_{p\in\mathsf{P}} \mathrm{d}^{\mathrm{disc}}_p$. In this case the linear discretization ind is said to be *split grading* for the tessellar differential module and ind induces a $\mathbb{Z}$-grading on the associated tessellar homology. The $\mathbb{Z}$-grading given by ind can be useful is some cases e.g. cellular homology and the treatment of tessellar homology for Morse pre-orders, cf. Sect.'s 4.3 – 4.4.

The advantage of using Betti numbers is that one can treat the $\mathbb{Z}$-grading given by ind as a grading on the Betti numbers. We start with using the $\mathbb{Z}$-grading on singular homology: $H^{\mathrm{disc}}(X) = \bigoplus_{q\in\mathbb{Z}} H^{\mathrm{disc}}_q(X)$. Define the *bi-graded Betti numbers* as: $\beta^{\mathrm{disc}}_{p,q} = \operatorname{rank} G_pH^{\mathrm{disc}}_q(X)$. The double *tessellar Poincaré polynomial* of X is defined as

$$P^{\mathrm{disc}}_{\lambda,\mu}(X) = \sum_{p,q\in\mathbb{Z}} \beta^{\mathrm{disc}}_{p,q}\lambda^p\mu^q. \tag{4.16}$$

The latter satisfies a variation on the standard *Morse relations*. Define $\beta^{\mathrm{disc}}_{p,q}(\xi) = \operatorname{rank} G_pH^{\mathrm{disc}}_q(G_{[\xi]}X)$ and associated Poincaré polynomial

$$P^{\mathrm{disc}}_{\lambda,\mu}\big(G_{[\xi]}X\big) = \sum_{p,q\in\mathbb{Z}} \beta^{\mathrm{disc}}_{p,q}(\xi)\lambda^p\mu^q.$$

The singular homology grading yields the splitting $\mathrm{d}^r_p = \bigoplus_{q\in\mathbb{Z}} \mathrm{d}^r_{p,q}$.

THEOREM 4.18 (Bi-graded Morse relations). *Let $X \xrightarrow{\mathrm{disc}} \mathfrak{X} \xrightarrow{\mathrm{ind}} \mathbb{Z}$ be a natural linear discretization. Then,*

$$\sum_{[\xi]\subset\mathfrak{X}} P^{\mathrm{disc}}_{\lambda,\mu}\big(G_{[\xi]}X\big) = P^{\mathrm{disc}}_{\lambda,\mu}(X) + \sum_{r=1}^{\infty}(1+\lambda^r\mu)Q^r_{\lambda,\mu}, \tag{4.17}$$

where $Q^r_{\lambda,\mu} = \sum_{p,q\in\mathbb{Z}}(\operatorname{rank}\operatorname{im}\mathrm{d}^r_{p+r,q+1})\lambda^p\mu^q \geqslant 0$. The sum over r is finite.

PROOF. In terms of the spectral sequences we have that $\mathscr{E}^0 = \bigoplus_{p,q\in\mathbb{Z}} \mathscr{E}^0_{p,q}$ and $\mathscr{E}^1 = \bigoplus_{p,q\in\mathbb{Z}} \mathscr{E}^1_{p,q}$ are isomorphic. For the spectral sequence we have the short exact sequences

$$0 \longrightarrow \ker \mathrm{d}^r_{p,q} \longrightarrow \mathscr{E}^r_{p,q} \longrightarrow \operatorname{im}\mathrm{d}^r_{p,q} \longrightarrow 0, \quad \text{and}$$
$$0 \longrightarrow \operatorname{im}\mathrm{d}^r_{p+r,q+1} \longrightarrow \ker \mathrm{d}^r_{p,q} \longrightarrow \mathscr{E}^{r+1}_{p,q} \longrightarrow 0.$$

[15] In Appendix C.2 the entries of d are explained.

the implies the relation

$$\operatorname{rank} \mathscr{E}^r_{p,q} = \operatorname{rank} \mathscr{E}^{r+1}_{p,q} + \operatorname{rank} \operatorname{im} \mathrm{d}^r_{p,q} + \operatorname{rank} \operatorname{im} \mathrm{d}^r_{p+r,q+1}.$$

Define the double Poincaré polynomials $P_{\lambda,\mu}(\mathscr{E}^r) := \sum_{p,q\in\mathbb{Z}}(\operatorname{rank} \mathscr{E}^r_{p,q})\lambda^p\mu^q$ and $Q^r_{\lambda,\mu} = \sum_{p,q\in\mathbb{Z}}(\operatorname{rank} \operatorname{im} \mathrm{d}^r_{p+r,q+1})\lambda^p\mu^q$. Then, the Poincaré polynomials satisfy $P_{\lambda,\mu}(\mathscr{E}^r) = P_{\lambda,\mu}(\mathscr{E}^{r+1}) + (1+\lambda^r\mu)Q^r_{\lambda,\mu}$. Iterating the above identities for $P_{\lambda,\mu}(\mathscr{E}^r)$ and using the fact that the spectral sequence converges yields Equation (4.17). □

Using bi-graded tessellar Betti numbers will prove to be very useful in setting up a more refined theory of spectral matrices. In Sect. 5.4.2 we exploit this idea in the setting of parabolic flows. This approach is reminiscent of the detailed connection matrix in [4]. If we again ignore the natural grading of singular homology the Morse relations will be

$$\sum_{[\xi]\subset\mathfrak{X}} P^{\mathrm{disc}}_\lambda\big(G_{[\xi]}X\big) = P^{\mathrm{disc}}_\lambda(X) + \sum_{r=1}^{\infty}(1+\lambda^r)Q^r_\lambda, \tag{4.18}$$

which is obtained by setting $\mu = 1$. Note that the property for ind to be split grading is that $Q^r_\lambda = 0$ for $r \geqslant 2$. The maximal value for r in (4.18) can be utilized to further coarsen ind in order to obtain a linear discretization that is split grading.

REMARK 4.19. A similar procedure to bi-graded tessellar Betti numbers can be followed for the $\mathfrak{X}/_\sim$-grading by using spectral systems, cf. [48].

REMARK 4.20. We do not refer to $C^{\mathrm{disc}}(X) = \bigoplus_{p\in\mathbb{Z}} G_pC^{\mathrm{disc}}(X)$ as the skeletal differential module since it is a coarsening of the tessellar differential module. The differential is obtained by coarsening the information. This issue comes up again in the next section.

4.3. Cellular homology

Let X be a finite CW-complex, i.e. a compact Hausdorff space that admits a CW-decomposition map $\mathrm{cell}\colon X \twoheadrightarrow \mathfrak{X}$, where $(\mathfrak{X}, \leqslant)$ is the poset of CW-cells with the face partial order. The *cellular differential module*, or *cellular chain complex*, denoted C^{cell}, is constructed according to the theory in Section 4.2. This coincides with the classical construction, as we outline below.

From the definition of CW-decomposition we have the composition

$$\begin{tikzcd} X \arrow[r, two heads, "\mathrm{cell}"] \arrow[rr, bend right, "\mathrm{skel}"'] & \mathfrak{X} \arrow[r, "\dim"] & \mathbb{N} \end{tikzcd}$$

which is a coarsening of the discretization cell. Since $\dim$ is order-preserving and since cell is a natural discretization, the composite discretization skel is natural and linear and thus continuous. Therefore skel defines a $\mathscr{T}$-consistent (not surjective) linear discretization of X. In the traditional set-up the cellular homology of X is defined in terms of the natural discretization map skel. As before define the filtering $\downarrow p \mapsto \mathrm{skel}^{-1}\downarrow p =: F_{\downarrow p}X$ with skeletal chain complex $C^{\mathrm{skel}}(X) :=$

$\bigoplus_{p\in\mathbb{N}} H\big(F_{\downarrow p}X, F_{\downarrow(p-1)}X\big)$.[16] The sets $G_pX = F_{\downarrow p}X \smallsetminus F_{\downarrow(p-1)}X$ are a disjoint union of p-cells $|\xi|$ in X and the homology is given by $H\big(F_{\downarrow p}X, F_{\downarrow(p-1)}X\big) \cong \bigoplus_{\xi\in G_p\mathfrak{X}} R\langle\xi\rangle$, where $G_p\mathfrak{X} = \dim^{-1} p$ and R is a principal ideal domain. Since the homologies $H\big(F_{\downarrow p}X, F_{\downarrow(p-1)}X\big)$ are free Theorem 4.7 yields a differential (strict spectral matrix) $\mathrm{d}^{\mathrm{skel}}\colon C^{\mathrm{skel}}(X) \to C^{\mathrm{skel}}(X)$ such that $H^{\mathrm{skel}}(X) \cong H(X)$. We will explain this construction now in a more detailed way.

REMARK 4.21. The above construction is the traditional way of constructing cellular homology for a finite CW-complex X. The $\mathbb{N}$-grading is special in the sense that if H is the singular homology functor and $\dim$ plays the role in ind in Sect. 4.2.3, then $G_pH_q(X) \neq 0$ if and only if $p = q$. Since all components are homeomorphic the skeletal chain complex is given by $C^{\mathrm{skel}}_q(X) = \bigoplus_{\dim(\xi)=q} R\langle\xi\rangle$ with boundary operator $\mathrm{d}^{\mathrm{skel}}_q$ with $\mathrm{d}^{\mathrm{skel}} = \bigoplus \mathrm{d}^{\mathrm{skel}}_q$. The linear discretization skel is split grading. Even though skel is natural this condition is not needed since the order is linear.

A more detailed way to treat the cellular complex is to use the face partial order $(\mathfrak{X}, \leqslant)$. As in the more general tessellar case we define the cellular chain complex[17] by

$$C^{\mathrm{cell}}(X) = \bigoplus_{\xi\in\mathfrak{X}} G_\xi C^{\mathrm{cell}}(X), \quad G_\xi C^{\mathrm{cell}}(X) = H\big(F_{\downarrow\xi}X, F_{\downarrow\xi^{\blacktriangleleft}}X\big) \cong H^{\mathrm{BM}}(G_\xi X),\ ^{18}$$

where $G_\xi X = |\xi|$ and its Borel-Moore homology of $G_\xi X$ is given by $H^{\mathrm{BM}}(G_\xi X) \cong R$. The module C^{cell} is a special case of the tessellar module for the discretization map cell. The filtering $\mathcal{U} \mapsto F_{\mathcal{U}}X$, $\mathcal{U} \in \mathsf{O}(\mathfrak{X}, \leqslant)$, defined by cell^{-1}, consists of good pairs and yields a chain generated, excisive Cartan-Eilenberg system $\mathbf{E}^{\mathrm{cell}}$ as outlined in Sections 4.1.3 and 4.2. Since $G_\xi C^{\mathrm{cell}}(X) \cong R$ the system $\mathbf{E}^{\mathrm{cell}}$ is finitely generated. We follow the procedure of Section 4.2. Theorem 4.7 yields the existence of (strict)

$$\mathrm{d}^{\mathrm{cell}}\colon C^{\mathrm{cell}}(X) \to C^{\mathrm{cell}}(X),$$

such that $\mathbf{E}\big(C^{\mathrm{cell}}, \mathrm{d}^{\mathrm{cell}}\big) \cong \mathbf{E}^{\mathrm{cell}}$. In particular, $H^{\mathrm{cell}}(X) \cong H(X)$ and

$$H^{\mathrm{cell}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'}X) \cong H(F_{\mathcal{U}}X, F_{\mathcal{U}'}X) \cong H^{\mathrm{BM}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'}X),\ ^{19} \tag{4.19}$$

for all $\mathcal{U}, \mathcal{U}' \in \mathsf{O}(\mathfrak{X}, \leqslant)$, $\mathcal{U}' \subset \mathcal{U}$. The pair $\big(C^{\mathrm{cell}}, \mathrm{d}^{\mathrm{cell}}\big)$ is the *cellular chain complex*. For the differential $\mathrm{d}^{\mathrm{cell}}$ we express the strict upper-triangular structure by $\mathrm{d}^{\mathrm{cell}}(\xi, \xi')\colon G_{\xi'}C^{\mathrm{cell}} \to G_\xi C^{\mathrm{cell}}$ and $\mathrm{d}^{\mathrm{cell}}(\xi, \xi') \neq 0$ implies that $\xi < \xi'$. The following lemma follows from choice of singular homology in the definition of $\mathbf{E}^{\mathrm{cell}}$.

LEMMA 4.22. *If* $\mathrm{d}^{\mathrm{cell}}(\xi, \xi') \neq 0$, *then* ξ' *covers* ξ.[20]

[16] By Remark 4.12 we also have $H\big(F_{\downarrow p}X, F_{\downarrow(p-1)}X\big) \cong H^{\mathrm{BM}}(G_pX)$.

[17] Even though we do not utilize the $\mathbb{Z}$-grading of singular homology we refer to C^{cell} as chain complex as opposed to differential module since the construction is based on the singular chain complex.

[18] Here $\downarrow\xi^{\blacktriangleleft} := (\downarrow\xi)^{\blacktriangleleft}$ denotes the immediate predecessor of $\downarrow\xi$.

[19] For latter isomorphism on Borel-Moore homology, cf. [28, 9].

[20] In a finite partial order ξ *covers* ξ if $\xi < \xi'$ and $[\xi, \xi'] = \{\xi, \xi'\}$. The pair $\{\xi, \xi'\}$ is a *covering pair*.

PROOF. For the singular chain complex the connecting homomorphism k_q in

$$\dashrightarrow H_q(F_{\downarrow\xi^\blacktriangleleft}X) \xrightarrow{i_q} H_q(F_{\downarrow\xi}X) \xrightarrow{j_q} H_q(F_{\downarrow\xi}X, F_{\downarrow\xi^\blacktriangleleft}X) \xrightarrow{k_q} H_{q-1}(F_{\downarrow\xi^\blacktriangleleft}X) \dashrightarrow$$

is degree -1 which implies that $\mathrm{d}^{\mathrm{cell}}$ is also degree -1 with respect to the $\mathbb{Z}$-grading of singular homology, i.e. $\mathrm{d}^{\mathrm{cell}}(\xi,\xi') = \bigoplus \mathrm{d}^{\mathrm{cell}}_q(\xi,\xi')$ and $\mathrm{d}^{\mathrm{cell}}_q(\xi,\xi')\colon G_{\xi'}C^{\mathrm{cell}}_q \to G_\xi C^{\mathrm{cell}}_{q-1}$, cf. [21]. For the cellular chain groups it holds that

$$G_\xi C^{\mathrm{cell}}_q(X) = H_q\big(F_{\downarrow\xi}X, F_{\downarrow\xi^\blacktriangleleft}X\big) \cong \begin{cases} R & \text{if } q = \dim\xi; \\ 0 & \text{if } q \neq \dim\xi, \end{cases}$$

Therefore, $\mathrm{d}^{\mathrm{cell}}_q(\xi,\xi') = 0$ unless $\dim\xi' = q$ and $\dim\xi = q-1$. □

In contrast to the general construction of Section 4.2, all of the nonzero entries $\mathrm{d}^{\mathrm{cell}}(\xi,\xi')$ are determined by the octahedral diagrams, i.e. the rolled out middle triangle in (4.4) (homology braid). Indeed, for a covering pair $\xi < \xi'$, with $\dim\xi' = q$, we choose a triple of down-sets (closed sets)

$$\downarrow\xi'^\blacktriangleleft \smallsetminus \xi \subset \downarrow\xi'^\blacktriangleleft \subset \downarrow\xi',$$

which yields $\mathrm{d}^{\mathrm{cell}}_q(\xi,\xi')\colon G_{\xi'}C^{\mathrm{cell}}_q \to G_\xi C^{\mathrm{cell}}_{q-1}$ given by the composition[21]

$$H_q\big(F_{\downarrow\xi'}X, F_{\downarrow\xi'^\blacktriangleleft}X\big) \xrightarrow{k_q} H_{q-1}\big(F_{\downarrow\xi'^\blacktriangleleft}X\big) \longrightarrow H_{q-1}\big(F_{\downarrow\xi'^\blacktriangleleft}X, F_{\downarrow\xi'^\blacktriangleleft\smallsetminus\xi}X\big). \tag{4.20}$$

where $G_{\xi'}C^{\mathrm{cell}}_q = H_q\big(F_{\downarrow\xi'}X, F_{\downarrow\xi'^\blacktriangleleft}X\big)$ and $G_\xi C^{\mathrm{cell}}_{q-1} = H_{q-1}\big(F_{\downarrow\xi'^\blacktriangleleft}X, F_{\downarrow\xi'^\blacktriangleleft\smallsetminus\xi}X\big)$. By the excisive property this construction is independent of the triple $\mathcal{U} \subset \mathcal{U}' \subset \mathcal{U}''$ with $\xi = \mathcal{U}' \smallsetminus \mathcal{U}$ and $\xi' = \mathcal{U}'' \smallsetminus \mathcal{U}'$.

Per Section 4.2.3 we consider the order-preserving map $\dim\colon \mathfrak{X} \to \mathbb{N}$ which plays the role of ind in order to obtain an $\mathbb{N}$-grading of H^{cell}. For the composition skel the sets $G_pX := \mathrm{skel}^{-1}p$ are convex. By Lemma 4.22 the differential $\mathrm{d}^{\mathrm{cell}}$ acts on $G_pC^{\mathrm{cell}}(X)$ as

$$\mathrm{d}^{\mathrm{cell}}\colon G_pC^{\mathrm{cell}}_p(X) \to G_pC^{\mathrm{cell}}_{p-1}(X).$$

If we write the restriction to $G_pC^{\mathrm{cell}}(X)$ as $\mathrm{d}^{\mathrm{cell}}_p$ then $\big(G_pC^{\mathrm{cell}}, \mathrm{d}^{\mathrm{cell}}_p\big)$ is a chain complex and $\dim$ is split grading for cellular homology. As a consequence $\dim$ yields the natural $\mathbb{N}$-grading of cellular homology.

REMARK 4.23. As in Section 4.2.2 we can define the standard Lefschetz complex and incidence numbers from the cellular homology.

REMARK 4.24. For the spectral sequence in Theorem 4.18 for $\dim$ we have that $\mathrm{d}^r_{p,q} = 0$ for $r \geqslant 2$ and therefore $Q^r_{\lambda,\mu} = 0$ for $r \geqslant 2$. Moreover, all homologies $\mathscr{E}^r_{p,q} = 0$ for $p \neq q$ and $r \leqslant 2$. For the Morse relations this implies

$$P_{\lambda\mu}(C^{\mathrm{cell}}) = P_{\lambda,\mu}(C^{\mathrm{cell}}) = P^{\mathrm{cell}}_{\lambda,\mu}(X) + (1+\lambda\mu)Q^1_{\lambda,\mu} = P^{\mathrm{cell}}_{\lambda\mu}(X) + (1+\lambda\mu)Q^1_{\lambda\mu}.$$

[21] In Franzosa's connection matrix theory these entries are called *flow defined*.

Moreover, $P_{\lambda\mu}(C^{\mathrm{cell}}) = \sum_{\xi\in\mathfrak{X}} P^{\mathrm{cell}}_{\lambda\mu}(\xi)$, with $P^{\mathrm{cell}}_{\lambda\mu}(\xi) = (\lambda\mu)^q$ and $q = \dim\xi$. This yields

$$\sum_{\xi\in\mathfrak{X}} (\lambda\mu)^{\dim\xi} = P^{\mathrm{cell}}_{\lambda\mu}(X) + (1+\lambda\mu)Q^1_{\lambda\mu},$$

which retrieve the standard Morse relations. The latter also follows if we use the fact that $\big(C^{\mathrm{cell}}, \mathrm{d}^{\mathrm{cell}}\big)$ is a chain complex.

4.4. Composite gradings and the homology for Morse pre-orders

The most important objective in chapter is to build an homology theory for the discretization map $\mathrm{tile}\colon X \to \mathsf{SC}$. In the first sections of this chapter we utilized Cartan-Eilenberg systems to discretize the algebraic topological information for arbitrary topological spaces. In this section we outline how discretization can be employed in a bi-topological setting. The objective is to use the factorization so that we can discretize two topologies: the space topology and the block-flow topology. To do so one can factor the two topologies in $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$ in different ways. For example $(X,\mathscr{T}) \to (\mathfrak{X}, \leqslant) \to (\mathfrak{X}, \leqslant^{\dagger}) \to (\mathsf{SC}, \leqslant)$, or $(X, \mathscr{T}_{\blacksquare}^{-}) \to (\mathfrak{X}, \leqslant_{\blacksquare}^{-}) \dashrightarrow (\mathfrak{X}, \leqslant^{\dagger}) \to (\mathsf{SC}, \leqslant)$, cf. Diagram (2.19). One can also invoke to the topology $\mathscr{T}^{\dagger}$ in this setting. In a more general setting the bi-topological discretization of algebraic invariants can be organized via the following diagram continuous maps

$$\begin{tikzcd}
(X, \mathscr{T}, \mathscr{T}') \arrow[r, bend left, "\mathrm{disc}"] \arrow[r, bend right, dashed, "\mathrm{disc}"'] & (\mathfrak{X}, \leqslant^{\dagger}) \arrow[r, "\mathrm{part}"] & (\mathsf{P}, \leqslant),
\end{tikzcd}$$

where $(X, \mathscr{T}, \mathscr{T}')$ is a bi-topological space, $(\mathfrak{X}, \leqslant^{\dagger})$ an antagonistic pre-order and $(\mathsf{P}, \leqslant)$ a finite poset. The factorization via $(\mathfrak{X}, \leqslant)$ and $(\mathfrak{X}, \leqslant')$ is given by Diagram (2.19). Recall,

$$\begin{tikzcd}
 & (\mathfrak{X}, \leqslant) \arrow[d, "\mathrm{id}"] \arrow[dr, "\mathrm{part}"] & \\
(X, \mathscr{T}, \mathscr{T}') \arrow[ur, "\mathrm{disc}"] \arrow[r, bend left, "\mathrm{disc}"] \arrow[r, bend right, dashed, "\mathrm{disc}"'] \arrow[dr, "\mathrm{disc}"'] & (\mathfrak{X}, \leqslant^{\dagger}) \arrow[r, "\mathrm{part}"] & (\mathsf{P}, \leqslant) \\
 & (\mathfrak{X}, \leqslant') \arrow[u, dashed, "\mathrm{id}"] \arrow[ur, dashed, "\mathrm{part}"'] &
\end{tikzcd}$$

We illustrate the discretization by considering one of the two topologies and the associated factorization:

$$\begin{tikzcd}
(X, \mathscr{T}) \arrow[r, two heads, "\mathrm{disc}"] \arrow[rrr, bend right, "\mathrm{tile}"'] & (\mathfrak{X}, \leqslant) \arrow[r, two heads, "\pi"] \arrow[rr, bend left, "\mathrm{part}"] & (\mathfrak{X}/_{\sim}, \leqslant) \arrow[r, "\varpi"] & (\mathsf{P}, \leqslant)
\end{tikzcd} \tag{4.21}$$

As before the discretization map tile yields a chain-generated Cartan-Eilenberg system $\mathbf{E}^{\mathrm{tile}}$, cf. Sect. 4.1.3 and Sect. 4.2.1. Under the assumption that disc is natural and yields finitely generated tessellar homology the same applies to $\mathbf{E}^{\mathrm{tile}}$. Define, using the singular chain complex for $(X, \mathscr{T})$,

$$C^{\mathrm{tile}}(X) := \bigoplus_{p \in \mathsf{P}} G_p C^{\mathrm{tile}}(X), \tag{4.22}$$

where $G_p C^{\mathrm{tile}}(X) := H\big(F_{\downarrow p} X, F_{\downarrow p^{\blacktriangleleft}} X\big)$ if the latter is a free R-module, or else choose a free differential module $\big(G_p C^{\mathrm{tile}}(X), \mathrm{d}\big)$ such that

(i) $H\big(G_p C^{\mathrm{tile}}, \mathrm{d}\big) \cong H\big(F_{\downarrow p} X, F_{\downarrow p^{\blacktriangleleft}} X\big)$;
(ii) $G_p C^{\mathrm{tile}}(X) \cong R^{s_p + 2r_p}$, cf. Defn. 4.8(ii).

By Theorem 4.7 there exists an $\mathsf{O}(\mathsf{P})$-filtered differential

$$\mathrm{d}^{\mathrm{tile}} \colon C^{\mathrm{tile}}(X) \to C^{\mathrm{tile}}(X)$$

such that $\mathbf{E}\big(C^{\mathrm{tile}}, \mathrm{d}^{\mathrm{tile}}\big) \cong \mathbf{E}^{\mathrm{tile}}$, cf. Sect. 4.1.2. In particular, $H^{\mathrm{tile}}(X) \cong H(X, \mathscr{T})$. The tessellar homology H^{tile} can also be defined for $(X, \mathscr{T}')$ and for $(X, \mathscr{T}^{\dagger})$. These are different invariants for the discretization $\mathrm{tile} \colon (X, \mathscr{T}, \mathscr{T}') \to (\mathsf{P}, \leqslant)$. Since $\mathrm{disc} \colon (X, \mathscr{T}) \to (\mathfrak{X}, \leqslant)$ is a natural discretization map we can utile the tessellar homology of disc. Consider the Cartain-Eilenberg system $\mathbf{E}^{\mathrm{tile}}$ where the E-terms are defined using the tessellar homology H^{disc} which isomorphic to the singular homology, i.e. $H^{\mathrm{disc}}(G_p X) \cong H\big(F_{\downarrow p} X, F_{\downarrow p^{\blacktriangleleft}} X\big)$. The tessellar homology for disc is defined via a $\mathfrak{X}/_\sim$-graded tessellar module $C^{\mathrm{disc}}(X) = \bigoplus_{[\xi] \in \mathfrak{X}/_\sim} G_{[\xi]} C^{\mathrm{disc}}(X)$. Convex sets in P are convex sets in $\mathfrak{X}$ and we define $G_p C^{\mathrm{tile}}(X)$ as $H^{\mathrm{disc}}(G_p X)$ is the latter is free. Otherwise choose $G_p C^{\mathrm{tile}}(X)$ as explained above. In the case of field coefficient in $\mathbb{K}$ the homology H^{disc} is free and there exists a strict P-graded differential module $\big(C^{\mathrm{disc}}(X), \mathrm{d}^{\mathrm{disc}}\big)$, cf. Theorem 4.7 and cf. [58].

REMARK 4.25. In the case we use field coefficients $R = \mathbb{K}$ the algorithm CONNECTIONMATRIX [31, Algorithm 6.8], which is based on algebraic-discrete Morse theory, takes as input $\big(C^{\mathrm{disc}}(X), \mathrm{d}^{\mathrm{disc}}\big)$ and outputs a strict P-graded differential module $\big(C^{\mathrm{tile}}(X), \mathrm{d}^{\mathrm{tile}}\big)$ which is $\mathsf{O}(\mathsf{P})$-filtered chain equivalent to the differential module $\big(C^{\mathrm{disc}}(X), \mathrm{d}^{\mathrm{disc}}\big)$ via $\mathsf{O}(\mathsf{P})$-filtered chain maps $h \colon C^{\mathrm{tile}}(X) \to C^{\mathrm{disc}}(X)$ and $h' \colon C^{\mathrm{disc}}(X) \to C^{\mathrm{tile}}(X)$. The $\mathsf{O}(\mathsf{P})$-filtered chain equivalence h induces isomorphisms $\big\{h_{\beta \smallsetminus \alpha} \colon H^{\mathrm{tile}}(G_{\beta \smallsetminus \alpha} X) \to H^{\mathrm{disc}}(G_{\beta \smallsetminus \alpha} X) \mid \alpha, \beta \in \mathsf{O}(\mathsf{P})\big\}$ which form an isomorphism of Cartan-Eilenberg systems:[22]

$$\begin{tikzcd}
H^{\mathrm{disc}}(G_{\beta\smallsetminus\alpha}X) \arrow[r] & H^{\mathrm{disc}}(G_{\gamma\smallsetminus\alpha}X) \arrow[r] & H^{\mathrm{disc}}(G_{\gamma\smallsetminus\beta}X) \arrow[r] & H^{\mathrm{disc}}(G_{\beta\smallsetminus\alpha}X) \\
H^{\mathrm{tile}}(G_{\beta\smallsetminus\alpha}X) \arrow[r] \arrow[u, "h_{\beta\smallsetminus\alpha}"] & H^{\mathrm{tile}}(G_{\gamma\smallsetminus\alpha}X) \arrow[r] \arrow[u, "h_{\gamma\smallsetminus\alpha}"] & H^{\mathrm{tile}}(G_{\gamma\smallsetminus\beta}X) \arrow[r] \arrow[u, "h_{\gamma\smallsetminus\beta}"] & H^{\mathrm{tile}}(G_{\beta\smallsetminus\alpha}X) \arrow[u, "h_{\beta\smallsetminus\alpha}"]
\end{tikzcd}$$

for all $\alpha, \beta, \gamma \in \mathsf{O}(\mathsf{P})$.

[22] cf. [21, Eqn. (1.2)].

REMARK 4.26. There is a special case when disc is a quasi-isomorphism, i.e.

$$H(\mathrm{disc})\colon H(X) \to H(\mathfrak{X})$$

induces an isomorphism $H(X, \mathscr{T}) \cong H(\mathfrak{X}, \leqslant)$, where the latter is taken to be the singular homology of the finite topological space $(\mathfrak{X}, \leqslant)$. This is a situation which commonly arises in practice, e.g. disc is a CW decomposition map and $(\mathfrak{X}, \leqslant)$ is a simplicial or cubical complex.[23] In this case we interpret (4.21) as

$$\begin{tikzcd}
(X, \mathscr{T}) \arrow[rr, "\mathrm{disc}", "\cong"'] \arrow[dr, "\mathrm{tile}"'] & & (\mathfrak{X}, \leqslant) \arrow[dl, "\mathrm{part}"] \\
& (\mathsf{P}, \leqslant) &
\end{tikzcd}$$

Let $C^{\mathrm{part}}(\mathfrak{X})$ be the tessellar P-graded differential module for part, for which $\mathbf{E}^{\mathrm{part}}$ is isomorphic to $\mathbf{E}^{\mathrm{tile}}$ given by the P-graded differential module $C^{\mathrm{tile}}(X)$ (this follows from an elementary five lemma argument), q.v. Sect. 6.2 for further discussion.

After this general interlude of bi-topological discretzation we return to the case $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$. Assume that $(X, \mathscr{T})$ is a finite regular CW-complex and let $\mathrm{disc} = \mathrm{cell}$, $\mathrm{part} = \mathrm{dyn} = \varpi$ and $\mathsf{P} = \mathsf{SC} = \mathfrak{X}/_{\sim^\dagger}$, cf. (4.24). If we use the discrete space $(\mathfrak{X}, \leqslant^\dagger)$ then the convex sets in SC are given by $\mathcal{U} \smallsetminus \mathcal{U}'$, $\mathcal{U}, \mathcal{U}' \in \mathsf{O}(\mathfrak{X}, \leqslant^\dagger)$. The homology $H^{\mathrm{part}}(\mathcal{U} \smallsetminus \mathcal{U}')$ is well-defined and isomorphic to $H^{\mathrm{tile}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'} X)$ per Remark 4.26. The homology $H^{\mathrm{tile}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'} X)$ is given by the relative homology $H(F_{\mathcal{U}} X, F_{\mathcal{U}'} X)$ and can be computed from a cellular chain complex. Summarizing, we have:

THEOREM 4.27. *Given the composition* $X \xrightarrow{\mathrm{cell}} \mathfrak{X} \xrightarrow{\mathrm{dyn}} \mathsf{SC}$. *Then,*

$$H^{\mathrm{dyn}}(\mathcal{U} \smallsetminus \mathcal{U}') \cong H^{\mathrm{tile}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'} X) \cong H^{\mathrm{cell}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'} X) \cong H^{\mathrm{BM}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'} X), \tag{4.23}$$

where $H^{\mathrm{BM}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'} X)$ *is the Borel-Moore homology of the Morse tile* $G_{\mathcal{U}\smallsetminus\mathcal{U}'} X$.

The homologies $H^{\mathrm{tile}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'} X)$ are invariants of a Morse pre-order $(\mathfrak{X}, \leqslant^\dagger)$. We can visualize this structure by augmenting the di-graph for $(\mathsf{SC}, \leqslant)$ by the homology $H^{\mathrm{part}}(\mathcal{S}) \cong H^{\mathrm{BM}}(G_{\mathcal{S}} X)$ at the nodes of the graph, cf. Fig. 1.5. The associated Cartan-Eilenberg system yields the homologies $H^{\mathrm{tile}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'} X)$. Such invariants can also be defined using the topologies $\mathscr{T}_{\blacksquare}^{-}$ and $\mathscr{T}^\dagger$, cf. Sect. 4.5. In the forthcoming sections we will implement these ideas for a large class of flows, so-called parabolic flows, and produce algebraic-combinatorial representations of

[23]Or more generally, if each f_ξ is a homeomorphism; such a CW complex is called regular.

its global dynamics.

$$
\begin{tikzcd}
 & (\mathfrak{X}, \leqslant) \arrow[d, "\mathrm{id}"] \arrow[dr, "\mathrm{dyn}"] & \\
(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-}) \arrow[ur, "\mathrm{cell}"] \arrow[r, bend left, "\mathrm{cell}"] \arrow[r, dashed, bend right, "\mathrm{cell}"] \arrow[dr, "\mathrm{cell}"'] & (\mathfrak{X}, \leqslant^{\dagger}) \arrow[r, "\mathrm{dyn}"] & (\mathsf{SC}, \leqslant) \\
 & (\mathfrak{X}, \leqslant_{\blacksquare}^{-}) \arrow[u, dashed, "\mathrm{id}"] \arrow[ur, dashed, "\mathrm{dyn}"'] &
\end{tikzcd}
\tag{4.24}
$$

REMARK 4.28. The map $\mathrm{tile}\colon X \to \mathsf{SC}$ is a discretization map and the homology H^{tile} is the tessellar homology of the discretization map tile. The two stage approach in this section allows us to compute H^{tile} from an SC-graded cellular chain complex. For any discretization $\mathrm{ind}\colon \mathsf{SC} \to \mathbb{Z}$ we can consider H^{tile} as a bi-graded homology theory (in the case of field coefficients) as described in Section 4.2.3. The bi-graded Betti numbers/homology depend on our choice of the discretization ind. A possible choice for ind is a linear extension of SC. In Section 5.5 we consider tessellar homology for parabolic flows with a linear discretization.

Consider the diagram

$$
\begin{tikzcd}
(X, \mathscr{T}) \arrow[r, "\mathrm{cell}"] \arrow[rr, Rightarrow, bend right, "\mathrm{tile}"'] \arrow[rrr, bend right=40, "\mathrm{skel}"'] & (\mathfrak{X}, \leqslant) \arrow[r, "\mathrm{dyn}"] & (\mathsf{SC}, \leqslant) \arrow[r, "\mathrm{ind}"] & \mathbb{Z}
\end{tikzcd}
$$

where we treat tile as discretization map and $\mathrm{ind}\colon \mathsf{SC} \to \mathbb{Z}$ is a linear discretization map. Then for for any convex set $\mathcal{U} \smallsetminus \mathcal{U}'$ the Morse relations in Thoerem 4.18 are given by

$$
\sum_{\mathcal{S} \in \mathcal{U} \smallsetminus \mathcal{U}'} P_{\lambda,\mu}^{\mathrm{tile}}(G_{\mathcal{S}} X) = P_{\lambda,\mu}^{\mathrm{tile}}(G_{\mathcal{U} \smallsetminus \mathcal{U}'} X) + \sum_{r=1}^{\infty} (1 + \lambda^r \mu) Q_{\lambda,\mu}^{r}. \tag{4.25}
$$

The poset $(\mathsf{SC}, \leqslant)$ yields a partial order on the pairs $\big(\mathcal{S}, P_{\lambda,\mu}^{\mathrm{tile}}(G_{\mathcal{S}} X)\big)$, $\mathcal{S} \in \mathsf{SC}$, via

$$
\big(\mathcal{S}, P_{\lambda,\mu}^{\mathrm{tile}}(G_{\mathcal{S}} X)\big) \leqslant \big(\mathcal{S}', P_{\lambda,\mu}^{\mathrm{tile}}(G'_{\mathcal{S}} X)\big) \quad \Longleftrightarrow \quad \mathcal{S} \leqslant \mathcal{S}'. \tag{4.26}
$$

The poset of pair $\big(\mathcal{S}, P_{\lambda,\mu}^{\mathrm{tile}}(G_{\mathcal{S}} X)\big)$ is denoted by $(\amalg, \leqslant^{\dagger})$ and is called the *tessellar phase diagram* of the Morse pre-order $(\mathfrak{X}, \leqslant^{\dagger})$. Considering only non-trivial Poincaré polynomials yields the *pure tessellar phase diagram* $(\overline{\amalg}, \leqslant^{\ddagger})$, with order-embedding

$$
(\overline{\amalg}, \leqslant^{\ddagger}) \hookrightarrow (\amalg, \leqslant^{\dagger}), \tag{4.27}
$$

cf. Fig. 1.4. The treatment of semi-flows up to this point is a tale of two topologies. In that setting the p-index in $\beta_{p,q}^{\mathrm{tile}}$ can be thought of as a manifestation of the topology induced by the semi-flow and the q-index as a manifestation of the space topology.

4.5. Morse tessellations, Morse decompositions and the Conley index

For a Morse pre-order $(\mathfrak{X}, \leqslant^\dagger)$ we have a Morse tessellation as given in (3.14). As a matter of fact the filtering $\mathcal{U} \mapsto F_{\mathcal{U}}X$, $\mathcal{U} \in \mathsf{O}(\mathfrak{X}, \leqslant^\dagger)$ defines a finite, distributive lattice $\mathsf{N} = \{F_{\mathcal{U}}X \mid \mathcal{U} \in \mathsf{O}(\mathfrak{X}, \leqslant^\dagger)\} \subset \mathsf{ABlock}(\varphi)$ of attracting blocks. This yields the Morse tessellation $(\mathsf{T}, \leqslant)$ given by

$$\mathsf{T}(\mathsf{N}) := \{T = U \smallsetminus U^{\blacktriangleleft} \mid U \in \mathsf{J}(\mathsf{N})\}, \quad T \leqslant T' \iff U \subset U', \tag{4.28}$$

where $T = G_{\mathcal{S}}X$, $U = F_{\downarrow\mathcal{S}}X$ and $U^{\blacktriangleleft} = F_{\downarrow\mathcal{S}\smallsetminus\mathcal{S}}X$, $\mathcal{S} \in \mathsf{SC}$, cf. [44]. Such a tessellation does model the 'direction' of dynamics but not the invariant dynamics. However, the compactness of the phase space $(X, \mathscr{T})$ does imply the existence of key invariant sets: attractors. Recall that a set $A \subset X$ is an *attractor* if there exists an attracting block U such that $A = \omega(U)$. The attractors of φ form a bounded, distributive lattice $\mathsf{Att}(\varphi)$ with binary operations $A \vee A' = A \cup A'$ and $A \wedge A' = \omega(A \cap A')$. The map $U \mapsto \omega(U)$ is a surjective lattice homomorphism, cf. [42]. By compactness:

$$U \in \mathsf{ABlock}(\varphi),\ U \neq \varnothing,\ \implies A = \omega(U) \neq \varnothing.$$

In terms of the sublattice N we obtain a sublattice $\omega\colon \mathsf{N} \twoheadrightarrow \mathsf{A} \subset \mathsf{Att}(\varphi)$, where $\mathsf{A} := \{A \in \mathsf{Att}(\varphi) \mid A = \omega(U),\ U \in \mathsf{N}\}$. In general this map need not be a isomorphism which yields an inportant conclusion: knowing N provides no insight into structure of A from information given by $(\mathfrak{X}, \leqslant^\dagger)$. However, topology can partly answer this question. From the map $\omega\colon \mathsf{N} \twoheadrightarrow \mathsf{A}$ we have the following congruence relation: $U \sim U'$ if and only if $\omega(U) = \omega(U')$. Since we cannot utilize this relation solely on the information given by $(\mathfrak{X}, \leqslant^\dagger)$ we use a topological principle for flows also known as *Wazewski's principle*,[24] cf. [12, Sect. II.2]. This can be restated as follows:

$$U \sim U' \quad \implies \quad H(U, U \cap U') \cong H(U \cup U', U') \cong 0, \tag{4.29}$$

where H is the singular homology functor. The key representation of dynamics is via a *tessellated Morse decomposition* which we define as the dual of the homomorphism $\omega\colon \mathsf{N} \twoheadrightarrow \mathsf{A}$. By Birkhoff duality we obtain an injective order preserving map $\mathsf{J}(\omega)\colon \mathsf{J}(\mathsf{A}) \hookrightarrow \mathsf{J}(\mathsf{N})$. Invoking the Conley form we obtain an injection $\pi\colon \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$, where the poset $(\mathsf{M}(\mathsf{A}), \leqslant)$, with $\mathsf{M}(\mathsf{A}) := \{M = A - A^{\blacktriangleleft} \mid A \in \mathsf{J}(\mathsf{A})\}$, is called a *Morse representation*. The notation $A - A^{\blacktriangleleft} := A \cap (A^{\blacktriangleleft})^* \neq \varnothing$ is the Conley form on $\mathsf{Att}(\varphi)$, cf. [44]. As before $M \leqslant M'$ if and only if $A \subset A'$. *The Morse sets in a Morse representation are never the empty set*! Via Birkhoff duality we can given an explicit formula for the embedding π. However, from [44] there exists a left inverse to π: the unique image $M \mapsto T$ satisfies $\mathrm{Inv}(T) = M$, where $\mathrm{Inv}(T)$ is the maximal invariant sets in T. The latter provides a easy way to construct the embedding π. As before we do not have control over the poset set $\mathsf{M}(\mathsf{A})$. Consider a Morse set M and $\pi(M) = T$. Then, for any pair $U, U' \in \mathsf{N}$ with $U \smallsetminus U' = \pi(M)$ we have

[24] To establish Wasewski's principle the continuity of φ is used a crucial way.

$\mathrm{Inv}(U \smallsetminus U') = \omega(U) - \omega(U') = M \neq \varnothing$ and $U \not\sim U'$. For the latter we can now invoke Wazewski's principle in (4.29):

$$H(U, U \cap U') \cong H(U \cup U', U') \neq 0 \quad \Longrightarrow \quad M = \mathrm{Inv}(U \smallsetminus U') \neq \varnothing. \tag{4.30}$$

By construction $H^{\mathrm{tile}}(G_{\mathcal{U} \smallsetminus \mathcal{U}'} X) \cong H(F_{\mathcal{U}} X, F_{\mathcal{U}'} X)$ only depends on $\mathcal{U} \smallsetminus \mathcal{U}'$ which justifies the definition

$$HC(T) := H^{\mathrm{tile}}(G_{\mathcal{U} \smallsetminus \mathcal{U}'} X), \quad T = F_{\mathcal{U}} X \smallsetminus F_{\mathcal{U}'} X, \tag{4.31}$$

and is called the *Conley index* of a Morse tile T, cf. [12]. In the case of a CW-decomposition via cell the Conley index T is given as the Borel-Moore homology of the tile T. The Conley index is an algebraic invariant for congruent pairs (U, U'). By construction $HC(T) \neq 0$ implies that T is the image of a Morse set M under π, i.e. $M = \mathrm{Inv}(T) \neq \varnothing$. The Conley index in (4.31) is not only well-defined for tiles in $\mathsf{T}(\mathsf{N})$ but for any Morse tile $T = U \smallsetminus U'$ obtained from attracting blocks $U, U' \in \mathsf{N}$.[25]

The algebraic topological approach in this chapter is to use invariants based on the topology $(X, \mathscr{T})$. Wazewski's principle allows an interpretation of the invariants that yield information about the invariant dynamics of the flow which defines the second topology $(X, \mathscr{T}_{\blacksquare}^{-})$. In the application of the theory to dynamical systems the topology $(X, \mathscr{T})$ is assumed to be given while the second topology $(X, \mathscr{T}_{\blacksquare}^{-})$ is not known a priori. However, for the latter we have information about discretizations $(\mathfrak{X}, \leqslant_{\blacksquare}^{-})$. This track of combining information of two topologies and invoking Wazewski's principle yields a powerful algebraic topological tool for studying invariant sets of flows. In Section 4.4 we also indicated that the tessellar homology can be defined for three topologies. The case outlined above is worked out. Invariants based on $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^{-})$ and $(\mathfrak{X}, \leqslant^{\dagger})$ take into account the bi-topological nature of the problem. In future work we will examine these more detailed algebraic topological invariants. For the application in the forthcoming chapter and applications to Conley theory the appoach in Section 4.4 suffices.

[25] The above arguments apply to sublattices of N such as $\varnothing \subset U \cap U' \subset U \subset X$, and $\varnothing \subset U' \subset U \cup U' \subset X$.

CHAPTER 5

Parabolic recurrence relations and flows

In the final part of this text we study a class of flows for which we demonstrate the discretization of topology and dynamics as described in the preceding chapters. The class of systems we consider are called *discrete parabolic flows*. Discrete parabolic flows and associated parabolic recurrence relations occur in various applications of dynamical systems and represent important classes of conservative dynamics as well as dissipative dynamics, cf. [66, 65, 67]. Examples of parabolic recurrence relations are discretizations of uniform parabolic PDE's, monotone twist maps and fourth order conservative differential equations, etc, cf. [25, 24]. The nature of discrete parabolic flows makes them very suitable for displaying the theory developed in this paper. The application to parabolic systems entails the definition of explicit CA-discretizations and MA-discretizations. The algebraic methods reveal a new invariant for braids and parabolic flows, q.v. Sect. 5.5.2.

5.1. Discretized braid diagrams

Braids can be treated in various ways. One way is to regard braids as a path in a two dimensional configuration space. The more hands-on way to think of braids as a collection of 'strands' between to copies of the Eucledian plane. A generic projection onto the strip $\mathbb{R}\times[0,1]$ contains all information by tagging intersections as postive, or negative. This representation is called a *braid diagram*. In this text we consider a special class braid diagrams, piecewise linear and with positive intersections. From [24] we recall the notion of closed *discretized braid daigram*.

DEFINITION 5.1. The space of *closed discretized period d braid diagrams* on m strands, denoted $\mathscr{D}^d_m$, is the space of unordered collections of *strands* $x = \{x^\alpha\}_{\alpha=0}^{m-1}$, defined as follows:

(i) (Strands): each strand $x^\alpha = (x^\alpha_0, x^\alpha_1, \ldots, x^\alpha_d) \in \mathbb{R}^{d+1}$ consists of $d+1$ *anchor points* x^α_j;
(ii) (Periodicity): $x^\alpha_d = x_0^{\theta(\alpha)}$ for all $\alpha = 1, \ldots, m$, for some permutation $\theta \in S_m$ (symmetric group);
(iii) (Non-degeneracy): for any pair of distinct strands x^α and $x^{\alpha'}$ such that $x^\alpha_i = x^{\alpha'}_i$ for some i, the following *transversality* condition holds

$$\big(x^\alpha_{i-1} - x^{\alpha'}_{i-1}\big)\big(x^\alpha_{i+1} - x^{\alpha'}_{i+1}\big) < 0.$$

The elements $x \in \mathscr{D}_m^d$ are referred to as *discretized braids*, or *discretized braid diagrams*. The spaces $\mathscr{D}_m^d$ can be topologized as metric spaces, cf. [24], and the connected components of $\mathscr{D}_m^d$, called *discrete braid classes*, are denoted by $[x]$.

REMARK 5.2. Note that the space $\mathscr{D}_1^d$ consists of tuples $x = (x_0, \cdots, x_d)$, with $x_0 = x_d$ and no additional conditions on x. Therefore, $\mathscr{D}_1^d \cong \mathbb{R}^d$ as metric spaces.

Let us start with an important invariant of discretized braids. Given a discretized braid $x \in \mathscr{D}_m^d$. Two of its strands x^α and $x^{\alpha'}$ *intersect* if

(i) $\big(x_i^\alpha - x_i^{\alpha'}\big)\big(x_{i+1}^\alpha - x_{i+1}^{\alpha'}\big) < 0$ for some i, or
(ii) if for some i, $x_i^\alpha = x_i^{\alpha'}$ and $\big(x_{i-1}^\alpha - x_{i-1}^{\alpha'}\big)\big(x_{i+1}^\alpha - x_{i+1}^{\alpha'}\big) < 0$.

Define $\iota(x^\alpha, x^{\alpha'}) := \#\big\{\text{number of intersection between } x^\alpha \text{ and } x^{\alpha'}\big\}$ as the local intersection number. Define the *crossing number* by

$$\boldsymbol{\lambda}(x) := \frac{1}{2} \sum_{\alpha \neq \alpha'} \iota(x^\alpha, x^{\alpha'}) \in \mathbb{N}.$$

The crossing number is an invariant for a braid classes $[x]$, i.e. $\boldsymbol{\lambda}$ is constant on components $[x] \subset \mathscr{D}_m^d$.

REMARK 5.3. Generically a discrete braid has the property $x_i^\alpha \neq x_i^{\alpha'}$ for all i and for all $\alpha \neq \alpha'$. The local intersection number can be defined for generic braids by counting sign changes, i.e. indices i for which (i) is satisfied.

Unordered sets $x = \{x^\alpha\}_{\alpha=1}^m$ for which Definition 5.1(iii) is not satisfied are called *singular braids* and are denoted by Σ_m^d. Pairs of strands for which Definition 5.1(iii) is not satisfied are called *non-transverse* and the crossing number is not defined in this case. We can however consider a variation on the crossing number that is defined for both discretized braids and singular braids. Let $x \in \Sigma_m^d$ be a singular braid. Then, following [22], we define the set

$$\mathscr{S}_\epsilon(x) := \Big\{\tilde{x} \in \mathscr{D}_m^d \mid |\tilde{x}_i^\alpha - x_i^\alpha| < \epsilon,\ \forall i \text{ and } \forall \alpha\Big\} \neq \varnothing.$$

The crossing numbers

$$\boldsymbol{\lambda}_-(x) := \min_{\mathscr{S}_\epsilon(x)} \boldsymbol{\lambda} \quad \text{and} \quad \boldsymbol{\lambda}_-(x) := \max_{\mathscr{S}_\epsilon(x)} \boldsymbol{\lambda}$$

are independent of ϵ provided $\epsilon > 0$ is sufficiently small and therefore are well-defined. For $x \in \mathscr{D}_m^d$ the numbers $\boldsymbol{\lambda}_-(x)$ and $\boldsymbol{\lambda}_+(x)$ are also defined in which case $\boldsymbol{\lambda}_-(x) = \boldsymbol{\lambda}_+(x) = \boldsymbol{\lambda}(x)$, cf. [22].

For a discrete braid x its *braid components* are given by the cycles of the permutation τ. The the orders of the cycles which we refer to as the *cycle orders* is another invariant for a braid class $[x]$. For example for $\theta \in S_5$ given by $\theta = (01)(234)$ the cycle orders are 2 and 3. We can define a special space of braids by coloring components. In this text we are interested in particular in braids with dual coloring. The space of *2-colored discretized braids* $\mathscr{D}_{n,m}^d$ consists of ordered pairs (x, y), where $x = \{x^\alpha\}_{\alpha=0}^{n-1} \in \mathscr{D}_n^d$ and $y = \{y^\beta\}_{\beta=0}^{m-1} \in \mathscr{D}_m^d$, and (x, y) satisfies Definition 5.1(i)-(iii). In other words $(x, y) \in \mathscr{D}_{n+m}^d$. We denote a 2-colored discretized braid by $x \text{ rel } y$. The canonical projection $\pi\colon \mathscr{D}^d_{n,m} \to \mathscr{D}^d_m$ given by $x \text{ rel } y \mapsto y$ yields the fibers $\mathscr{D}^d_n \text{ rel } y = \pi^{-1}(y)$. A connected component $[x] \text{ rel } y$ is called a *(discrete) relative braid class component* with skeleton y. A connected component $[x \text{ rel } y]$ in $\mathscr{D}^d_{n+m}$ is called a *(discrete) relative braid class*. The fibers $\pi^{-1}(y) \cap [x \text{ rel } y]$ consist of relative braid class components $[x] \text{ rel } y$ and are referred to as the *(discrete) relative braid class fibers* of $[x \text{ rel } y]$. In most situations a braid class fiber consists of a single braid class component.

DEFINITION 5.4. A relative braid class $[x \text{ rel } y]$ is called *non-degenerate*, or *proper* if the cycle orders in x differ form the cycle orders in y. In particular, for $x \text{ rel } y \in \mathscr{D}^d_{1,m}$ is proper if the cycle orders in y are all larger than 1. In latter case the *skeleton* y is also called *proper*.

If $[x \text{ rel } y]$ is proper, then its fibers $\pi^{-1}(y) \cap [x \text{ rel } y]$ are and therefore also its components $[x] \text{ rel } y$ are proper as well. For a skeleton $y \in \mathscr{D}^d_m$ we consider the singular braids in $x \text{ rel } y \in \Sigma^d_{n,m}$. Denote the fiber of singular relative braids by $\Sigma^d_n \text{ rel } y = \pi^{-1}(y)$.

5.2. Parabolic flows

Discretized braids introduced in the previous section are intimately related to a class of recurrence relations.

DEFINITION 5.5. A *parabolic recurrence relation* (of period $d > 0$) is a system of equations of the form

$$R_i(x_{i-1}, x_i, x_{i+1}) = 0, \quad i \in \mathbb{Z}, \tag{5.1}$$

where each $R_i\colon \mathbb{R}^3 \to \mathbb{R}$ is a smooth function such that

(i) $R_{i+d} = R_i$ for all i;
(ii) $\partial_1 R_i > 0$ and $\partial_3 R_i > 0$.[1]

We denote the recurrence relation by $\mathbf{R} = (R_i)$.

The following proposition establishes periodic solutions of parabolic recurrence relations as discretized braid diagrams. At a latter stage we also define associated flows which yields an even stronger symbiosis between parabolic recurrence relations and discretized braids.

PROPOSITION 5.6 (cf. [24]). *Let $x = \{x^\alpha\}_{\alpha=0}^{m-1}$ be a set of strands satisfying Definition 5.1(i)-(ii) with the property that $R_i(x^\alpha_{i-1}, x^\alpha_i, x^\alpha_{i+1}) = 0$, for all i and all α. Then, $x \in \mathscr{D}^d_m$, i.e. Definition 5.1(iii) is also satisfied. Such a discretized braid is called a stationary braid with respect to* (5.1). *In particular, the crossing number of x is well-defined.*

[1]One can weaken the monotonicity with one of the inequalities to be $\geqslant$ for every i. For convenience in this paper we assume strict inequalities for both partial derivatives unless indicated explicitly.

Associated to a parabolic recurrence relation we consider the following system of differential equations:

$$\dot{x}_i = R_i(x_{i-1}, x_i, x_{i+1}), \quad i \in \mathbb{Z}. \tag{5.2}$$

The solution operator as well as the system of differential equations will be referred to as a *discrete parabolic flow*. The objective is to find k-periodic solutions of parabolic recurrence relations, i.e. sequences (x_i), with $x_{i+kd} = x_i$ for some $k \in \mathbb{N}$, which satisfies Equation (5.1). In order to build a suitable theory for periodic solutions we use the concept of discretized braids, cf. [24] Multiple periodic sequences with possible different periods may be regarded as a discretized braid diagram.

The parabolic equation in (5.2) defines a *local* flow φ on the space of 1-periodic sequences $\mathscr{D}_1^d \cong \mathbb{R}^d$ with the standard metric topology. We refer to φ as a *parabolic flow*. We say that a braid $y \in \mathscr{D}_m^d$ is a *skeletal braid* for φ if Equation (5.1) is satisfied for all $y^\alpha \in y$. Shorthand notation $\mathbf{R}(y) = 0$ and y is also referred as a *skeleton* for φ. Recall that a skeleton for which the cycle orders are all larger than 1 is called a *proper skeleton*. Relative braids $x \text{ rel } y \in \mathscr{D}_1^d \text{ rel } y$ for which y is proper are proper as relative braids. In this case there is an important relation between parabolic dynamics and the crossing number invariants.

PROPOSITION 5.7 (cf. [24], [22]). *Let $y \in \mathscr{D}_m^d$ be a proper skeleton (stationary braid) for φ, i.e. all cycle orders are strictly larger than 1, and let $x \text{ rel } y \in \Sigma_1^d \text{ rel } y$. Then, for $\epsilon > 0$ sufficiently small, we have that*

(i) $\varphi(t,x) \text{ rel } y \in \mathscr{D}_1^d \text{ rel } y$ *for all* $0 < |t| \leqslant \epsilon$*;*

(ii) for all $-\epsilon \leqslant t_- < 0 < t_+ \leqslant \epsilon$ *it holds that*

$$\begin{aligned}\boldsymbol{\lambda}_+(x \text{ rel } y) &= \boldsymbol{\lambda}\big(\varphi(t_-,x) \text{ rel } y\big)\\ &> \boldsymbol{\lambda}\big(\varphi(t_+,y) \text{ rel } y\big) = \boldsymbol{\lambda}_-(x \text{ rel } y).\end{aligned}$$

PROOF. Denote by $k\#y$ the k-fold covering of y, i.e. we take k concatenated copies of y. The braid y gives a permutation θ on the on the symbols $\{1, \cdots, m\}$, cf. Defn. 5.1(ii). Choose k to be order of the permutation θ. Then, $k\#y$ consists of 1-periodic sequences for $i \in \{0, \cdots, kd\}$. Moreover,

$$\boldsymbol{\lambda}\big(k\#x \text{ rel } k\#y\big) = k\boldsymbol{\lambda}\big(x \text{ rel } y\big). \tag{5.3}$$

Since $R_{i+d} = R_i$ we have that the flow φ^k generated by $\boldsymbol{\psi}$ on $\mathscr{D}_1^{kd}$ is given by the k-fold covering if we choose $x \in \mathscr{D}_1^d$, i.e. $\varphi^k(t, k\#x) = k\#\varphi(t,x)$. If $x \in \Sigma_1^d \text{ rel } y$, then $k\#x \text{ rel } k\#y \in \Sigma_1^{kd} \text{ rel } k\#y$. Let y^α be a strand such x and y^α are non-transverse. Since all cycle orders of y are larger than 1 we have that all relative braids $x \text{ rel } y$ are proper and thus $x - y^\alpha \neq 0$ for all α (strands don not coincide). By the main result in [22] this implies that $\iota\big(\varphi^k(t_-,x), k\#y^\alpha\big) > \iota\big(\varphi^k(t_+,x), k\#y^\alpha\big)$ for all $-\epsilon \leqslant t_- < 0 < t_+ \leqslant \epsilon$. If we combine this with the crossing number for k-fold coverings we obtain

$$\sum_\alpha \iota\big(\varphi^k(t_-,x), k\#y^\alpha\big) > \sum_\alpha \iota\big(\varphi^k(t_+,x), k\#y^\alpha\big),$$

which implies that $\boldsymbol{\lambda}\big(\varphi^k(t_-,x) \text{ rel } k\#y\big) > \boldsymbol{\lambda}\big(\varphi^k(t_+,x) \text{ rel } k\#y\big)$ and thus

$$\boldsymbol{\lambda}\big(k\#\varphi(t_-,x) \text{ rel } k\#y\big) > \boldsymbol{\lambda}\big(k\#\varphi(t_+,x) \text{ rel } k\#y\big).$$

Property (5.3) then gives

$$\boldsymbol{\lambda}\big(\varphi(t_-,x) \text{ rel } y\big) > \boldsymbol{\lambda}\big(\varphi(t_+,x) \text{ rel } y\big).$$

From [22, Thm. 1(ii)] it also follows that $\boldsymbol{\lambda}\big(\varphi(t_-,x) \text{ rel } y\big) = \boldsymbol{\lambda}_+(x \text{ rel } y)$ and $\boldsymbol{\lambda}\big(\varphi(t_+,x) \text{ rel } y\big) = \boldsymbol{\lambda}_-(x \text{ rel } y)$, which completes the proof □

As a consequence of the above proposition we conclude that $\boldsymbol{\lambda}_+(\cdot \text{ rel } y)$ is a discrete Lyapunov function for φ and the value of the Lyapunov function strictly drops at singular relative braids.

REMARK 5.8. If we do not require the cycle orders for y to be strictly larger than 1 then if x may coalesce with the 1-periodic strands in y in which case associated the singular braid is stationary and Proposition 5.7(ii) does not hold in that case. For application of these technique for improper braid classes recall [24, 67].

Besides crossing numbers and cycle orders another invariant for relative braid classes can be defined, cf. [24], and which is of algebraic topological nature. In this case we assume that the components $[x]$ rel y are bounded as sets in $\mathbb{R}^d$. In Section 5.5.1 we provided an extensive account of the algebraic invariant which is also referred to as the *Braid Conley index*. The theory in [24] implies that the invariants are homotopy invariants, i.e. a homotopy $h_s(y)$ in $\mathscr{D}^d_m$ yields isomorphic invariants. The latter is useful for choosing convenient representatives y for studying the relative braid class fibers in $\mathscr{D}^d_1 \text{ rel } y$.

5.3. Closure algebra discretizations for parabolic flows

In this section we assume that a skeleton $y \in \mathscr{D}^d_m$ always consist of two extremal strands: define $y^- = y^0$ and $y^+ = y^{m-1}$ such that the remaining strands satisfy $y_i^- < y_i^\alpha < y_i^+$, for all $\alpha = 1, \cdots, m-2$. The latter collection of strands is denoted by $\mathring{y}$. The skeleton y now induces a bounded cubical complex as will be explained in Section 5.3.1.

DEFINITION 5.9. Let $y \in \mathscr{D}^d_m$ be as described above and assume that $\mathring{y}$ is proper, cf. Defn. 5.4. We consider points x in

$$X = \big\{x \in \mathscr{D}^d_1 \mid y_i^- \leqslant x_i \leqslant y_i^+\big\} \subset \mathbb{R}^d, \tag{5.4}$$

which is a compact metric space with metric topology induced by $\mathbb{R}^d$, cf. Rem. 5.2. Let φ be a parabolic (local) flow on X generated by Equation (5.2) where the parabolic recurrence relation $\mathbf{R}$ satisfies $\mathbf{R}(y) = 0$.

Regard $\bar{y} = \{y^-, y^+\}$ as sub-skeleton. For x equal to either y^- or y^+ the local flow φ is stationary. If $x_i = y_i^-$, or $x_i = y^+$ for at least one i, i.e. $x \in \partial X$ as subset of $\mathbb{R}^d$, then $x \text{ rel } \bar{y} \in \Sigma^d_1 \text{ rel } \bar{y}$. Proposition 5.7 then implies that $\boldsymbol{\lambda}\big(\varphi(t,x) \text{ rel } \bar{y}\big) =$

$\boldsymbol{\lambda}(x \text{ rel } \bar{y}) = 0$, which yields $\varphi(t,x) \in X$ for all $t \geqslant 0$. If we invoke the remaining strands in y the semi-flow φ will display monotone behavior with respect to relative braid classes in the spirit of a Morse tessellation as is described in the forthcoming sections. In practical terms $\varphi(t,x)$ rel y is generically contained in a relative braid and $\varphi(t,x)$ rel y can evolve from one braid class to the next and not return. Summarizing, the parabolic flows satisfy the following properties:

(i) $\varphi\colon \mathbb{R}^+ \times X \to X$ is smooth semi-flow on X;
(ii) the braid $y \in \mathscr{D}^d_m$ is a skeleton for φ.

We now study parabolic flows on X for a fixed skeleton $y \in \mathscr{D}^d_m$. From this point on we assume that y is a skeleton as described above with $\mathring{y}$ proper.

REMARK 5.10. The boundary strands y^- and y^+ model boundary conditions on the parabolic flow pushing in. Using alternating strands we can also model boundary condition pushing out, cf. [24]. Variations on push-in or push-out boundary conditions can be obtained by different combinations of multi-strand braids. In this paper we restrict to push-in boundary conditions modeled via the strands y^- and y^+.

5.3.1. Discretization of the metric topology on X. We define a cubical complex $\mathfrak{X}$ for X as follows. From the definition of X there exists a natural grading on X via co-dimension of tangencies. Define

$$G_d X := \{x \mid x_i \neq y_i^{\alpha_i}, \forall i, \alpha\};$$
$$G_{d-k} X := \{x \mid x_{i_1} = y_{i_1}^{\alpha_{i_1}}, \cdots, x_{i_k} = y_{i_k}^{\alpha_{i_k}},\ i_1 \neq \cdots \neq i_k\},$$

where $k = 1, \cdots, d$. Note that the indices α_{i_j} are not necessarily distinct. If $\alpha = \alpha_{i_1} = \cdots = \alpha_{i_d}$, then $\alpha \in \{0, m-1\}$ by the definition of X and the assumption that $\mathring{y}$ is proper. The top cells $\xi \in \mathfrak{X}^\top =: G_d\mathfrak{X}$ label the connected components of the set $G_d X$, which we refer to as *generic braids*. The $(d-k)$-dimensional cells $\xi \in G_{d-k}\mathfrak{X}$ label the connected components of the set $G_{d-k}X$, cf. Fig. 5.3(a). All cells realize as open rectangular cuboids $|\xi|$ in X and are thus homeomorphic to open k-balls in $\mathbb{R}^k$, $k > 0$. The set $\mathfrak{X}$ of cells is forms a CW-decomposition of X, cf. Sect. 2.6.2. The map

$$\text{cell}\colon X \to \mathfrak{X},\ \text{defined by}\ \text{cell}(x) = \xi,\ \text{for}\ x \in |\xi|, \tag{5.5}$$

is a CW-decomposition map and thus Boolean. The co-dimension provides the dimension grading of $\mathfrak{X}$:

$$\dim\colon X \to \mathfrak{X},\ \text{with}\ \dim \xi = q \text{ if and only if } |\xi| \in G_q X.$$

The face partial order on $\mathfrak{X}$ defines the discrete closure operator $\text{cl}\, \xi = \downarrow \xi$. The triple $(\mathfrak{X}, \text{cl}, |\cdot|)$ is a CA-discretization for X and the finite algebra $(\mathsf{Set}(\mathfrak{X}), \text{cl})$ is the associated closure algebra discretezation of $(\mathsf{Set}(X), \text{cl})$.

Note that generically a relative braid x rel $y \in \mathscr{D}^d_1$ rel y is a point in a top cell $|\xi|$, $\xi \in \mathfrak{X}^\top$. This justifies the notation $\boldsymbol{\lambda}(\xi)$ as the crossing number of a top cell. The same applies to the crossing numbers $\boldsymbol{\lambda}_-$ and $\boldsymbol{\lambda}_+$, i.e. $\boldsymbol{\lambda}_-(\xi) := \min\{\boldsymbol{\lambda}(\eta) \mid$

$\eta \in \mathbf{star}\,\xi \cap \mathcal{X}^{\top}\}$ and $\boldsymbol{\lambda}_+(\xi) := \max\{\boldsymbol{\lambda}(\eta) \mid \eta \in \mathbf{star}\,\xi \cap \mathcal{X}^{\top}\}$. Define the following combined crossing number function on $\mathcal{X}$:

$$\boldsymbol{\Lambda}\colon \mathcal{X} \to \mathbb{N} \times \mathbb{N}, \quad \xi \mapsto \boldsymbol{\Lambda}(\xi) := \big(\boldsymbol{\lambda}_-(\xi), \boldsymbol{\lambda}_+(\xi)\big)$$

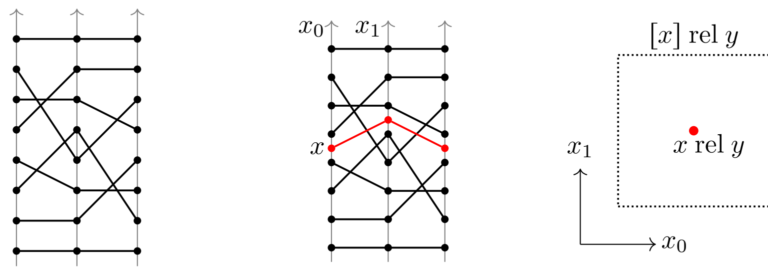

FIGURE 5.1. Skeleton y [left] and y with free strand x (in red) [middle]. A relative braid class component $[x]$ rel y which is a top-cell in $\mathcal{X}$ [right].

REMARK 5.11. It is often convenient to use the following normal form for the skeleton y: for any fixed i the cross-section $(y_i^0, y_i^1, \ldots, y_i^{m-1})$ is a permutation of $\{0, 1, \ldots, m-1\}$. That is, the anchor points (y_i^α) are integers and take unique values between 0 and $m-1$. This implies that the pairs (i, y_i^α) lay on the integer lattice within the box $[1, d+1] \times [0, m-1]$. The cubical complex $\mathcal{X}$ is comprised of cells ξ which we define as follows. Consider sets $I_i^1 = (i, i+1) \subset \mathbb{R}$ for for $i \in \{0, \cdots, m-2\}$ and sets $I_i^2 = \{i\}$, for $i \in \{0, \cdots, m-1\}$. A cell ξ is given as

$$|\xi| = I_{i_1}^{j_1} \times I_{i_2}^{j_2} \times \ldots \times I_{i_d}^{j_d}, \tag{5.6}$$

where $j_k \in \{1, 2\}$, $i_k \in \{0, \cdots, m-2\}$ for $I_{i_k}^1$ and $i_k \in \{0, \cdots, m-1\}$ for $I_{i_k}^2$. Figure 5.3(a) below shows a skeleton y in normal form and Figure 5.3(b) depicts the cubical complex consisting of unit squares. We can now use the cubical complex to describe a normal form of the braid component $[x]$ rel y with y in normal form. The dimension of a cell ξ is given by $\dim \xi = \sum_{j_k \neq 2} j_k$, where the integers are determined by the representation of $|\xi|$ in (5.6).

Figure 5.1[left] above is an example of a skeleton $y \in \mathscr{D}_8^2$. Figure 5.2[left] displays the CW-decomposition in terms of cubes. Figure 5.2[left] also gives the crossing numbers $\boldsymbol{\lambda}(\xi)$ for the top cells $\xi \in \mathcal{X}^{\top}$.

5.3.2. Discretization of flow topologies. In the previous section we described a natural CW-decomposition to discretize the metric topology on X. We now define a second discrete topology on $\mathcal{X}$ via a pre-order such that the map $\boldsymbol{\Lambda}\colon \mathcal{X} \to \mathbb{N} \times \mathbb{N}$ is order-preserving and which serves the purpose of discretizing the block-flow topology $\mathscr{T}_{\blacksquare}^{-}$. The order on $\mathbb{N} \times \mathbb{N}$ is the product order, i.e. $(a, b) \leqslant (a', b')$ if and only if $a \leqslant a'$ and $b \leqslant b'$. We build a pre-order for discretizing $\mathscr{T}_{\blacksquare}^{-}$ in two steps:

(i) Define the (symmetric) relation $\mathfrak{C} \subset \mathfrak{X} \times \mathfrak{X}$ as the *partial adjacency relation*

$$(\xi, \xi'), (\xi', \xi) \in \mathfrak{C} \iff \xi' \in \mathfrak{X}^{\top} \text{ and } \xi \in \mathrm{cl}\, \xi'.^{2}$$

(ii) Define the *discrete flow relation* $\boldsymbol{\psi} \subset \mathfrak{C}$ as follows:

$$(\xi, \xi') \in \boldsymbol{\psi} \iff \boldsymbol{\Lambda}(\xi) \leqslant \boldsymbol{\Lambda}(\xi')$$

By construction $\boldsymbol{\Lambda}$ is order-preserving. Observe that $(\xi, \xi) \in \boldsymbol{\psi}$ if and only if $\xi \in \mathfrak{X}^{\top}$. Since every relative braid and relative singular braid is associated to a unique cell ξ we devide up the cells in $\mathfrak{X}$ into two groups, the *regular cells* $\xi \in \mathcal{X}_{\mathrm{reg}}$ and the *singular cells* $\xi \in \mathcal{X}_{\mathrm{sing}}$ which may be characterized as follows:

$$\begin{aligned} \xi \in \mathcal{X}_{\mathrm{reg}}, &\quad \text{if and only if} \quad \boldsymbol{\lambda}(\xi) = \boldsymbol{\lambda}_{-}(\xi) = \boldsymbol{\lambda}_{+}(\xi); \\ \xi \in \mathcal{X}_{\mathrm{sing}}, &\quad \text{if and only if} \quad \boldsymbol{\lambda}_{-}(\xi) < \boldsymbol{\lambda}_{+}(\xi). \end{aligned}$$

The transitive closure of $\boldsymbol{\psi}$ yields the discrete derivative operator $\boldsymbol{\Gamma}^{+} := (\boldsymbol{\psi}^{+})^{-1}$ and the transitive, reflexive closure $\boldsymbol{\psi}^{+=}$ defines the pre-order $\leqslant^{+}$ and yields the discrete closure operator $\mathbf{cl}^{+} := (\boldsymbol{\psi}^{+=})^{-1}$, and $\mathbf{cl}^{+} = \mathrm{id} \cup \boldsymbol{\Gamma}^{+}$.

LEMMA 5.12. *The discrete closure operator* $\mathbf{cl}^{+}$ *defined above satisfies the continuity condition* $\mathrm{cl}^{+}|\mathcal{U}| \subset |\mathbf{cl}^{+}\mathcal{U}|$ *for all* $\mathcal{U} \subset \mathfrak{X}$.

PROOF. By Remark 3.16 it suffices to show that $\varphi(t,x) \in |\mathbf{cl}^{+}\xi|$ for all $t \geqslant 0$, for all $x \in |\xi|$ and for all $\xi \in \mathfrak{X}$. Let $\xi \in \mathfrak{X}$, then $\varphi(t,x) \in |\,\mathbf{star}\,\xi|$ for all $0 \leqslant t \leqslant \tau_x$ for some τ_x sufficiently small. If $\xi \in \mathcal{X}_{\mathrm{reg}}$, then $\mathbf{star}\,\xi \subset \mathbf{cl}^{+}\xi$[3] and if $\xi \in \mathcal{X}_{\mathrm{sing}}$, then $\mathbf{star}\,\xi \not\subset \mathbf{cl}^{+}\xi$. In the former case $\varphi(t,x) \in |\,\mathbf{star}\,\xi| \subset |\mathbf{cl}^{+}\xi|$ for all $0 \leqslant t \leqslant \tau_x$ for some τ_x sufficiently small. In the latter case we argue as follows. From Proposition 5.7 we have that $\varphi(-t,x) \in |\xi^{-}|$,[4] $\varphi(t,x) \in |\xi^{+}|$, for all $0 < t \leqslant \tau_x$, with $\xi^{-}, \xi^{+} \in \mathbf{star}\,\xi \cap \mathfrak{X}^{\top}$. Moreover, $\boldsymbol{\lambda}(\xi^{-}) = \boldsymbol{\lambda}_{+}(\xi)$ and $\boldsymbol{\lambda}(\xi^{+}) = \boldsymbol{\lambda}_{-}(\xi)$. This implies that

$$\boldsymbol{\Lambda}(\xi^{-}) > \boldsymbol{\Lambda}(\xi) = \big(\boldsymbol{\lambda}_{+}(\xi), \boldsymbol{\lambda}_{-}(\xi)\big) > \boldsymbol{\Lambda}(\xi^{+}). \tag{5.7}$$

This proves that $(\xi^{+}, \xi) \in \boldsymbol{\psi}$ and thus $\varphi(t,x) \in |\mathbf{cl}^{+}\xi|$ for all $0 \leqslant t \leqslant \tau_x$ for some τ_x sufficiently small.

The above arguments show that for every $x \in X$ the flow satisfies $\varphi(t,x) \in |\eta|$ for all $0 < t \leqslant \tau_x$ for some $\eta \in \mathcal{X}_{\mathrm{reg}}$. Let $t' > \tau_x$ be the first time that $\varphi(t',x) \in |\zeta|$ for some $\zeta \in \mathcal{X}_{\mathrm{sing}} \cap \mathbf{cl}\,\eta$. Then, by (5.7), $\boldsymbol{\Lambda}(\eta) > \boldsymbol{\Lambda}(\zeta)$ and thus $(\zeta, \eta) \in \boldsymbol{\psi}$ and consequently $\zeta \in \mathbf{cl}^{+}\xi$. We can repeat the above argument to conclude that the criterion in Remark 3.16 is satisfied for all $t \geqslant 0$. $\square$

REMARK 5.13. In view of Theorem 3.14, since X is compact, it suffices to prove that $\varphi(t,x) \in |\mathbf{cl}^{+}\xi|$ for all $0 \leqslant t \leqslant \tau_*$, some $\tau_* > 0$.

---

[2] In term of the face partial order this reads $\xi \leqslant \xi'$.

[3] Here $\mathbf{star}$ is associated with the discrete topology $(\mathfrak{X}, \leqslant)$ given by the CW-decomposition.

[4] A cell ξ^{-} does not exist when ξ corresponds to a boundary cell for X.

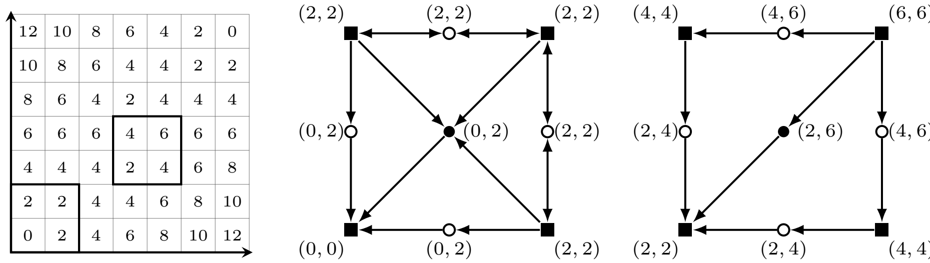

FIGURE 5.2. Cubical complex $\mathfrak{X}$ and values of $\boldsymbol{\lambda}(\xi)$, $\xi \in \mathfrak{X}^{\top}$ [left], for the braid diagram displayed in Fig. 5.3. The outlined regions are magnified in the middle and right figures and indicate the relation $\boldsymbol{\psi}$. On the various cells $\xi \in \mathfrak{X}$ it gives the Lyapunov function $\boldsymbol{\Lambda}$.

REMARK 5.14. Equation (5.7) in the above proof also implies that $(\xi, \xi^-) \in \boldsymbol{\psi}$ which is equivalent to $(\xi^-, \xi) \in \boldsymbol{\psi}^{-1}$. The latter implies $\varphi(-t, |\xi|) \subset |\mathbf{cl}^- \xi|$ for all $t \geqslant 0$ and for all $\xi \in \mathfrak{X}$, where $\mathbf{cl}^-$ is the closure operator obtained from the opposite relation $\boldsymbol{\psi}^{-1} \subset \mathfrak{X} \times \mathfrak{X}$.

Lemma 5.12 shows that $(\mathsf{Set}(\mathfrak{X}), \mathbf{cl}^+)$ is a CA-discretization for the Alexandrov topology $(X, \mathscr{T}^+)$ defined by the parabolic flow. If we also invoke the observation that $(\xi, \xi) \in \boldsymbol{\psi}$ if and only if $\xi \in \mathfrak{X}^{\top}$ we can prove an even stronger statement.

LEMMA 5.15. *The discrete derivative operator* $\boldsymbol{\Gamma}^+$ *satisfies* $\Gamma^+ |\mathcal{U}| \subset |\boldsymbol{\Gamma}^+ \mathcal{U}|$ *for all* $\mathcal{U} \subset \mathfrak{X}$.

PROOF. The proof follows along the same lines as Lemma 5.12. As pointed out in Remark 3.16 it suffices to show that $\varphi(t, x) \in |\boldsymbol{\Gamma}^+ \xi|$ for all $t > 0$, for all $x \in |\xi|$ and for all $\xi \in \mathfrak{X}$. If $\xi \in \mathcal{X}_{\rm reg}$, then either $\xi \in \mathfrak{X}^{\top}$, or $\boldsymbol{\Lambda}(\eta) = \boldsymbol{\Lambda}(\xi)$ for all all $\eta \in \mathbf{star}\, \xi$. If $\xi \in \mathfrak{X}^{\top}$, then $\varphi(t, x)$ stays in $|\xi|$ for all $0 \leqslant t \leqslant \tau_x$ for some τ_x sufficiently small and thus in $|\boldsymbol{\Gamma}^+ \xi|$ by the observation that $(\xi, \xi) \in \boldsymbol{\psi}$ if and only if $\xi \in \mathfrak{X}^{\top}$. The remaining case follows as before and therefore $\varphi(t, x) \in |\,\mathbf{star}\, \xi| \subset |\boldsymbol{\Gamma}^+ \xi|$ for all $0 \leqslant t \leqslant \tau_x$ for some τ_x sufficiently small. In particular for $0 < t \leqslant \tau_x$. The case $\xi \in \mathcal{X}_{\rm sing}$ follows as in the proof of Lemma 5.12. □

REMARK 5.16. Note that Lemma 5.15 implies Lemma 5.12. This indicates that the decomposition $\mathbf{cl}^+ = \mathrm{id} \cup \Gamma^+$ on $\mathsf{Set}(X)$ allows a discretization $\mathbf{cl}^+ = \mathrm{id} \cup \boldsymbol{\Gamma}^+$ on $\mathsf{Set}(\mathfrak{X})$ with an explicit derivative operator $\boldsymbol{\Gamma}^+$. The same conclusions hold for $\mathbf{cl}^- = \mathrm{id} \cup \boldsymbol{\Gamma}^-$ defined via the opposite relation $\boldsymbol{\psi}^{-1} \subset \mathfrak{X} \times \mathfrak{X}$.

The CA-discretizations for $(X, \mathscr{T}^-)$ and $(X, \mathscr{T}^+)$ are constructed according to the crossing number function $\boldsymbol{\Lambda}$ and are especially designed to display the behavior of φ with respect to singular braids, cf. Prop. 5.7. One can define more refined discretizations that yield more discrete forward invariant sets. The next lemma explains why we use this particular construction of CA-discretization for $(X, \mathscr{T}^-)$ and $(X, \mathscr{T}^+)$.

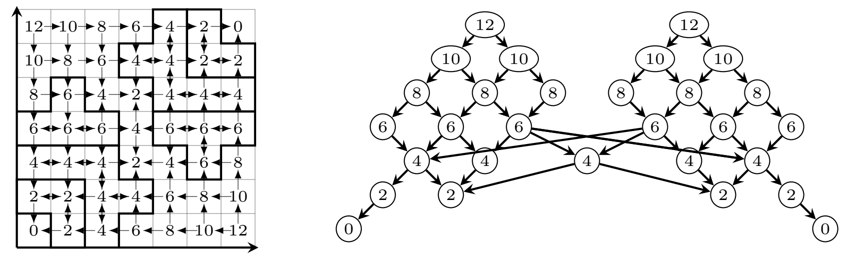

FIGURE 5.3. Cubical complex $\mathfrak{X}$, $\boldsymbol{\lambda}$ restricted to $\mathfrak{X}^\top$ and the closure operator $\mathbf{cl}^\top$ induced by the relation $\boldsymbol{\psi}$ visualized as a directed graph [left]. Elements of SC with more than one top cell are outlined. Poset SC, where vertices are labeled by $\boldsymbol{\lambda}$ [right].

LEMMA 5.17. *Let $\mathcal{U} \subset \mathfrak{X}$ be a closed, forward invariant set for $\boldsymbol{\psi}$, i.e. $\mathbf{cl}\,\mathcal{U} = \mathcal{U}$ and $\mathbf{cl}^+\mathcal{U} = \mathcal{U}$. Then,*

$$\boldsymbol{\Gamma}^+\mathcal{U} \subset \mathbf{int}\,\mathcal{U}. \tag{5.8}$$

In particular, $\mathcal{U}$ is a regular closed set in $\mathfrak{X}$.

PROOF. For $\xi \in \mathcal{U}$ we distinguish between $\xi \in \mathfrak{X}^\top$ and $\xi \notin \mathfrak{X}^\top$. We start with the latter. For $\xi \notin \mathfrak{X}^\top$ consider two cases: (i) $\xi \in \mathbf{int}\,\mathcal{U}$. Then, $\mathbf{star}\,\xi \subset \mathbf{int}\,\mathcal{U}$, and by the definition of $\boldsymbol{\psi}$ we have that $\boldsymbol{\psi}^{-1}[\xi] := \{\eta \mid (\eta,\xi) \in \boldsymbol{\psi}\} \subset \mathbf{star}\,\xi \subset \mathrm{int}\,\mathcal{U}$.[5] (ii) $\xi \in \mathcal{U} \smallsetminus \mathbf{int}\,\mathcal{U} = \mathbf{cl}\,\mathcal{U} \smallsetminus \mathbf{int}\,\mathcal{U} = \mathrm{bd}\,\mathcal{U}$ (using the fact that $\mathbf{cl}\,\mathcal{U} = \mathcal{U}$). Then, $\mathbf{star}\,\xi \not\subset \mathcal{U}$. Under the condition that $\mathbf{cl}^+\mathcal{U} = \mathcal{U}$ regular cells $\xi \in \mathcal{X}_{\mathrm{reg}}$ satisfy the property that $\mathbf{star}\,\xi \subset \mathcal{U}$. This implies that boundary cells are singular cells $\xi \in \mathcal{X}_{\mathrm{sing}}$. As before $\boldsymbol{\psi}^{-1}[\xi] \subset \mathbf{star}\,\xi \subset \mathfrak{X}^\top$ and thus $\boldsymbol{\psi}^{-1}[\xi]$ is open. Since $\mathbf{cl}^+\mathcal{U} \subset \mathcal{U}$ also $\boldsymbol{\psi}^{-1}[\xi] \subset \mathcal{U}$ which yields $\boldsymbol{\psi}^{-1}[\xi] \subset \mathfrak{X}^\top \cap \mathcal{U} \subset \mathbf{int}\,\mathcal{U}$.

Consider the case $\xi \in \mathcal{U}^\top = \mathcal{U} \cap \mathfrak{X}^\top$. Then, $\boldsymbol{\psi}^{-1}[\xi] \subset \mathcal{U} \cap \mathbf{cl}\,\xi \subset \mathcal{U}$ since $\mathbf{cl}\,\mathcal{U} = \mathcal{U}$ and $\mathbf{cl}^+\mathcal{U} = \mathcal{U}$. Let $\eta \in \boldsymbol{\psi}^{-1}[\xi] \smallsetminus \mathfrak{X}^\top$ be a cell which is not interior to $\mathcal{U}$, i.e. $\eta \in \mathrm{bd}\,\mathcal{U}$. By the same argument as before $\eta \in \mathcal{X}_{\mathrm{sing}}$ and by Equation (5.7) we have that $\boldsymbol{\Lambda}(\eta) > \boldsymbol{\Lambda}(\xi)$ which yields $(\xi,\eta) \in \boldsymbol{\psi}$ and $(\eta,\xi) \notin \boldsymbol{\psi}$. The latter contradicts the existence of boundary cells $\eta \in \boldsymbol{\psi}^{-1}[\xi]$. Consequently, $\boldsymbol{\psi}^{-1}[\xi] \subset \mathbf{int}\,\mathcal{U}$, which holds for every $\xi \in \mathcal{U}$.

Iterating this procedure gives $(\boldsymbol{\psi}^{-1})^k[\xi] \subset \mathbf{int}\,\mathcal{U}$, $k \geqslant 1$ and thus $\boldsymbol{\Gamma}^+\xi \subset \mathbf{int}\,\mathcal{U}$, cf. Thm. 3.14. This proves that $\boldsymbol{\Gamma}^+\mathcal{U} \subset \mathbf{int}\,\mathcal{U}$. The realization $|\mathcal{U}|$ is a closed, forward invariant set and satisfies $\varphi(t,|\mathcal{U}|) \subset \mathrm{int}\,|\mathcal{U}|$ for all $t > 0$. Therefore $|\mathcal{U}|$ is thus a closed attracting block. Such sets a regular closed by Theorem 3.23. Since the CA-discretization $(\mathfrak{X}, \leqslant, |\cdot|)$ is Boolean the same holds for the sets $\mathcal{U}$ in $(\mathfrak{X}, \leqslant)$. $\square$

From the previous consideration we propose a pre-order $\leqslant^\dagger$ and we show that $\leqslant^\dagger$ yields the right continuity properties with respect to cell as defined in (5.5).

[5] Note that this does not require closedness for neither of the discrete topologies.

Define the pre-order

(5.9) $$\leqslant^\dagger := \leqslant \vee \leqslant^+,$$ [6]

i.e. $\mathcal{U}$ is closed in $(\mathfrak{X}, \leqslant^\dagger)$ if and only if $\mathbf{cl}\,\mathcal{U} = \mathcal{U}$ and $\mathbf{cl}^+\mathcal{U} = \mathcal{U}$.

THEOREM 5.18. *The pre-order* $\leqslant^\dagger$ *defines a Morse pre-order for* $(X, \mathscr{T}, \mathscr{T}_{\blacksquare}^-)$*, i.e. the maps*

$$\mathrm{cell}\colon (X, \mathscr{T}) \to (\mathfrak{X}, \leqslant^\dagger), \quad \text{and} \quad \mathrm{cell}\colon (X, \mathscr{T}_{\blacksquare}^-) \to (\mathfrak{X}, \geqslant^\dagger),$$

are continuous.

PROOF. To proof the above statement we need to show that for $\mathcal{U} \subset \mathfrak{X}$ closed in $(\mathfrak{X}, \leqslant^\dagger)$ the pre-image $\mathrm{cell}^{-1}\mathcal{U} = |\mathcal{U}|$ is closed in $(X, \mathscr{T})$ and open in $(X, \mathscr{T}_{\blacksquare}^-)$. Let $\mathcal{U}$ be closed in $(\mathfrak{X}, \leqslant^\dagger)$. Then, $\mathbf{cl}\,\mathcal{U} = \mathcal{U}$ and $\mathbf{cl}^+\mathcal{U} = \mathcal{U}$ which implies that $|\mathcal{U}|$ is closed in $(X, \mathscr{T})$. By Lemma 5.17

$$\varphi(t, |\mathcal{U}|) \subset \bigcup_{t>0} \varphi(t, |\mathcal{U}|) = \mathbf{\Gamma}^+|\mathcal{U}| \subset |\mathbf{\Gamma}^+\mathcal{U}| \subset |\,\mathbf{int}\,\mathcal{U}| = \mathrm{int}\,|\mathcal{U}|,\ \forall t > 0,$$

which, by Lemma 3.7, proves that $|\mathcal{U}|$ is open in $(X, \mathscr{T}_{\blacksquare}^-)$. □

The above theorem shows that sets that are closed in both discrete topologies are closed attracting blocks and therefore $\leqslant \vee \leqslant^+ = \leqslant^\dagger$ defines a Morse pre-order for parabolic flows with skeleton y and which is an antagonistic coarsening of discretizations for both $\mathscr{T}$ and $\mathscr{T}_{\blacksquare}^-$, cf. Sect. 2.5, i.e. take $\leqslant^\dagger$ and $\geqslant^\dagger$ respectively.

REMARK 5.19. If $\mathcal{U} \subset \mathfrak{X}$ is a closed, backward invariant set for ψ, i.e. $\mathbf{cl}\,\mathcal{U} = \mathcal{U}$ and $\mathbf{cl}^-\mathcal{U} = \mathcal{U}$. Then, $\mathbf{\Gamma}^-\mathcal{U} \subset \mathbf{int}\,\mathcal{U}$. The arguments in the proof of Lemma 5.17 remain unchanged if we replace ψ with ψ^{-1}. For the latter we use the first part of the inequality in (5.7). This proves the same statement for closed, backward invariant sets.

REMARK 5.20. From the previous consideration we can define a discretization of the block-flow topology $\mathscr{T}_{\blacksquare}^-$. Using the relation ψ as defined above we have the derivative operators $\mathbf{\Gamma}^-, \mathbf{\Gamma}^+\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$. Define the discrete operator $\mathbf{\Gamma}_{\blacksquare}^- := \mathbf{\Gamma}^-\mathbf{cl}$. Since $\mathbf{\Gamma}^-$ gives a DA-discretization for the Alexandrov topology $\mathscr{T}^-$ we have by Remark 3.16 that

$$\varphi(-t, \mathbf{cl}\,|\xi|) = \varphi(-t, |\mathbf{cl}\,\xi|) \subset |\mathbf{\Gamma}^-\mathbf{cl}\,\xi| = |\mathbf{\Gamma}_{\blacksquare}^-\xi|, \quad \forall t > 0.$$

The discrete operator $\mathbf{\Gamma}_{\blacksquare}^-$ satisfies the hypotheses of Lemma 3.13 and therefore $\mathbf{cl}_{\blacksquare}^-\colon \mathsf{Set}(\mathfrak{X}) \to \mathsf{Set}(\mathfrak{X})$, given by Lemma 3.13 defines a CA-discretization for $(X, \mathscr{T}_{\blacksquare}^-)$. The antagonistic coarsening of $(\mathfrak{X}, \mathbf{cl}, \mathbf{cl}_{\blacksquare}^-, |\cdot|)$ is the pre-order $\leqslant^\dagger$ in (5.9).

The following lemma gives a characterization of the transitive, reflexive closure of ψ.

[6]The meet of two pre-orders is defined as the transitive closure of the union of the two relations, i.e. $\leqslant \vee \leqslant^+ := (\leqslant \cup \leqslant^+)^+$.

LEMMA 5.21. *$(\xi,\xi') \in \psi^{+=}$ if and only if there exist cells $\xi_0,\cdots\xi_\ell$, with $\xi_0 = \xi$, $\xi_\ell = \xi'$ and $(\xi_i,\xi_{i+1}) \in \mathfrak{X}^\top \times G_{d-1}\mathfrak{X}$, or $(\xi_i,\xi_{i+1}) \in G_{d-1}\mathfrak{X} \times \mathfrak{X}^\top$, such that*

$$(\xi_0,\xi_1),(\xi_1,\xi_2),\cdots,(\xi_{\ell-2},\xi_{\ell-1}),(\xi_{\ell-1},\xi_\ell) \in \psi. \tag{5.10}$$

PROOF. We prove the lemma in one direction since (5.10) trivially implies $(\xi,\xi') \in \psi^{+=}$. We may assume without loss of generality that $\xi,\xi' \in \mathfrak{X}^\top$, $\xi \neq \xi'$. Indeed, if $\xi \notin \mathfrak{X}^\top$ then $\xi \in \mathcal{X}_{\rm reg}$, or $\xi \in \mathcal{X}_{\rm sing}$. For the former we can choose $\tilde{\xi} \in \mathbf{star}\,\xi \cap \mathfrak{X}^\top$ such that $\mathbf{\Lambda}(\tilde{\xi}) = \mathbf{\Lambda}(\xi)$, and for the latter case $\boldsymbol{\lambda}_-(\xi) < \boldsymbol{\lambda}_+(\xi)$. Therefore we can choose $\tilde{\xi} \in \mathbf{star}\,\xi \cap \mathfrak{X}^\top$ such that $\big(\boldsymbol{\lambda}_-(\xi),\boldsymbol{\lambda}_-(\xi)\big) = \mathbf{\Lambda}(\tilde{\xi}) < \mathbf{\Lambda}(\xi) = \big(\boldsymbol{\lambda}_-(\xi),\boldsymbol{\lambda}_+(\xi)\big)$. We can thus choose $\tilde{\xi} \in \mathbf{star}\,\xi \cap \mathfrak{X}^\top$ such that $\mathbf{\Lambda}(\tilde{\xi}) \leqslant \mathbf{\Lambda}(\xi)$. Similarly, if $\xi' \notin \mathfrak{X}^\top$ we can choose $\tilde{\xi}' \in \mathbf{star}\,\xi' \cap \mathfrak{X}^\top$ such that $\mathbf{\Lambda}(\xi) \leqslant \mathbf{\Lambda}(\tilde{\xi}')$.

By definition $(\xi,\xi') \in \psi$ if (a) $\xi' \in \mathfrak{X}^\top$ and $\xi \in \mathbf{cl}\,\xi'$, or (b) $\xi \in \mathfrak{X}^\top$ and $\xi' \in \mathbf{cl}\,\xi$, or (c) $\xi = \xi'$ which we may exclude in the proof. Consequently, $(\xi,\xi') \in \psi^{+=}$ implies the existence of

$$(\xi,\eta_0),(\eta_0,\sigma_1),\cdots,(\sigma_{k-1},\eta_{k-1}),(\eta_{k-1},\xi') \in \psi,$$

with $\eta_j \notin \mathfrak{X}^\top$ and $\sigma_j \in \mathfrak{X}^\top$. The lemma is proved if we prove (5.10) for the case $(\sigma,\eta),(\eta,\sigma') \in \psi$, $\sigma,\sigma' \in \mathfrak{X}^\top$ and $\eta \notin \mathfrak{X}^\top$. Assume without loss of generality that $\eta \in G_k\mathfrak{X}$, $k < d-1$. By the definition of ψ it holds that $\mathbf{\Lambda}(\sigma) \leqslant \mathbf{\Lambda}(\eta) \leqslant \mathbf{\Lambda}(\sigma')$, which is equivalent to

$$\big(\boldsymbol{\lambda}(\sigma),(\boldsymbol{\lambda}(\sigma)\big) \leqslant \big(\boldsymbol{\lambda}_-(\eta),(\boldsymbol{\lambda}_+(\eta)\big) \leqslant \big(\boldsymbol{\lambda}(\sigma'),(\boldsymbol{\lambda}(\sigma')\big).$$

Since $\boldsymbol{\lambda}_-(\eta)$ is minimal over $\mathbf{star}\,\eta \cap \mathfrak{X}^\top$ and $\boldsymbol{\lambda}_+(\eta)$ is maximal over $\mathbf{star}\,\eta \cap \mathfrak{X}^\top$ it follows that $\boldsymbol{\lambda}(\sigma) = \boldsymbol{\lambda}_-(\eta) \leqslant \boldsymbol{\lambda}_+(\eta) = \boldsymbol{\lambda}(\sigma')$. Moreover, every $\sigma'' \in \mathbf{star}\,\eta \cap \mathfrak{X}^\top$ satisfies $\boldsymbol{\lambda}(\sigma) \leqslant \boldsymbol{\lambda}(\sigma'') \leqslant \boldsymbol{\lambda}(\sigma')$. Choose $\sigma'' \in \mathbf{star}\,\eta \cap \mathfrak{X}^\top$ such that $\mathbf{cl}\,\sigma \cap \mathbf{cl}\,\sigma'' \cap G_{d-1}\mathfrak{X} \neq \varnothing$ and let $\eta' \in G_{d-1}\mathfrak{X}$ be the unique cell in $\mathbf{cl}\,\sigma \cap \mathbf{cl}\,\sigma'' \cap G_{d-1}\mathfrak{X}$. Then, $\mathbf{\Lambda}(\sigma) \leqslant \mathbf{\Lambda}(\eta') \leqslant \mathbf{\Lambda}(\sigma'') \leqslant \mathbf{\Lambda}(\sigma')$ and $(\sigma,\eta'),(\eta',\sigma'') \in \psi$. Observe that $\mathbf{cl}\,\sigma'$ and $\mathbf{cl}\,\sigma''$ intersect in a cell $\tilde{\eta} \in \mathbf{star}\,\eta$ with $\mathbf{\Lambda}(\sigma'') \leqslant \mathbf{\Lambda}(\tilde{\eta}) \leqslant \mathbf{\Lambda}(\sigma')$ and $\mathbf{star}\,\tilde{\eta} \subset \mathbf{star}\,\eta$. Now repeat the above steps by using the cells in $\mathbf{star}\,\eta \cap \mathfrak{X}^\top$ at most once. This process terminates after finitely many steps, proving the lemma. □

THEOREM 5.22. *The partial equivalence classes of $\psi^{+=}$ correspond to the discrete relative braid class components in $\mathscr{D}^d_1$ rel y for a given skeleton y.*

PROOF. We distinguish regular and singular cells. Every regular cell ξ determines a discrete relative braid. This assignment is not one-to-one in general. Let the braid class component $[x]$ rel y be the connected component of x rel y for some $x \in |\xi|$. Any point x' rel $y \in [x]$ rel y corresponds to a cell $\xi' \in \mathcal{X}_{\rm reg}$, with $x \in |\xi'|$. Let $\gamma\colon [0,1] \to [x]$ rel y be a path joining x and x'. Then, $\gamma(s) \in |\xi_s| \subset [x]$ rel y, $\xi_s \in \mathcal{X}_{\rm reg}$ for all $s \in [0,1]$. If $\xi \in \mathcal{X}_{\rm reg}$, then $\mathbf{star}\,\xi \subset \mathcal{X}_{\rm reg}$ and $\mathbf{\Lambda}(\eta) = \mathbf{\Lambda}(\xi)$ for all $\eta \in \mathbf{star}\,\xi$. Therefore, the set $\bigcup_{s\in[0,1]} |\mathbf{star}\,\xi_s|$ is an open covering of γ. Since $\mathbf{\Lambda}(\mathbf{star}\,\xi_s)$ is constant for all s, the compactness of path γ implies that $\mathbf{\Lambda}$ is constant on $\bigcup_{s\in[0,1]} \mathbf{star}\,\xi_s$ and in particular $\mathbf{\Lambda}(\xi) = \mathbf{\Lambda}(\xi')$. The path also yields a chain ξ_i satisfying (5.10) which proves, using Lemma 5.21, that $(\xi,\xi') \in \psi^{+=}$. Since this holds for any two relative braids in $[x]$ rel y we conclude that the set of all cells $\mathcal{U}$ with $|\mathcal{U}| = [x]$ rel y is contained in a partial equivalence class of $\boldsymbol{\psi}^{+=}$. Conversely, equivalent cells belong to the same braid class component which proves that the braid class components are realized by the partial equivalence classes of $\boldsymbol{\psi}^{+=}$. □

From the theory in Section 3.4 we have that $\leqslant^\dagger$ restricted to $\mathfrak{X}^\top$ defines a condensed Morse pre-order $(\mathfrak{X}^\top, \leqslant^\top)$. The condensed Morse pre-order $(\mathfrak{X}^\top, \leqslant^\top)$ can be characterized as follows. We define a relation $\boldsymbol{\psi}^\top$ on $\mathfrak{X}^\top$ in two steps.

(i) Let $\boldsymbol{\mathcal{C}}^\top \subset \mathfrak{X}^\top \times \mathfrak{X}^\top$ be the (symmetric) *adjacency relation* given by

$$(\xi, \xi') \in \boldsymbol{\mathcal{C}}^\top \iff G_{d-1}\mathfrak{X} \cap \mathbf{cl}\,\xi \cap \mathbf{cl}\,\xi' \neq \varnothing \quad \text{and} \quad \xi \neq \xi',$$

i.e. $(\xi, \xi') \in \boldsymbol{\mathcal{C}}^\top$ if and only if the cells $\|\xi\|$ and $\|\xi'\|$ intersect along a $(d-1)$-dimensional face.[7]

(ii) Let $\boldsymbol{\psi}^\top \subset \boldsymbol{\mathcal{C}}^\top$ be defined as follows:

$$(\xi, \xi') \in \boldsymbol{\psi}^\top \iff \boldsymbol{\lambda}(\xi) \leqslant \boldsymbol{\lambda}(\xi'),$$

cf. Fig. 5.3[left] and [right].

THEOREM 5.23. *The transitive, reflexive closure $\boldsymbol{\psi}^{\top+=}$ is the restriction $\leqslant^\top$ of $\leqslant^\dagger$ to $\mathfrak{X}^\top$. In other words, $\boldsymbol{\psi}^{\top+=}$ is a condensed Morse pre-order for φ.*

PROOF. Let $(\xi, \xi') \in \boldsymbol{\psi}^\top$, then ξ and ξ' are adjacent top cells and $\boldsymbol{\lambda}(\xi) \leqslant \boldsymbol{\lambda}(\xi')$. By definition there exists a unique cell $\eta \in G_{d-1}\mathfrak{X} \cap \mathbf{cl}\,\xi \cap \mathbf{cl}\,\xi'$, and $\mathbf{star}\,\eta = \{\eta, \xi, \xi'\}$. The Lyapunov functions for these cells are given by $\boldsymbol{\Lambda}(\xi) = \big(\boldsymbol{\lambda}(\xi), \boldsymbol{\lambda}(\xi)\big)$, $\boldsymbol{\Lambda}(\eta) = \big(\boldsymbol{\lambda}(\xi), \boldsymbol{\lambda}(\xi')\big)$ and $\boldsymbol{\Lambda}(\xi') = \big(\boldsymbol{\lambda}(\xi'), \boldsymbol{\lambda}(\xi')\big)$. Consequently, $\boldsymbol{\Lambda}(\xi) \leqslant \boldsymbol{\Lambda}(\eta) \leqslant \boldsymbol{\Lambda}(\xi')$ and thus $(\xi, \eta), (\eta, \xi') \in \boldsymbol{\psi}$. By definition $(\xi, \xi') \in \boldsymbol{\psi}^{\top+=}$ is equivalent to pairs

$$(\xi_0, \xi_1), \cdots, (\xi_{\ell-1}, \xi_\ell) \in \boldsymbol{\psi}^\top,$$

with $\xi_0 = \xi$ and $\xi_k = \xi'$. Therefore, $(\xi, \xi') \in \boldsymbol{\psi}^{\top+=}$ implies $(\xi, \xi') \in \boldsymbol{\psi}^{+=}$. We obtain the inclusion, $\boldsymbol{\psi}^{\top+=} \subset \boldsymbol{\psi}^{+=} \cap (\mathfrak{X}^\top \times \mathfrak{X}^\top)$.

Conversely, let $(\xi, \xi') \in \boldsymbol{\psi}^{+=} \cap (\mathfrak{X}^\top \times \mathfrak{X}^\top)$. By Lemma 5.21 there exist $\xi_0, \cdots, \xi_{2\ell}$, with $\xi_0 = \xi$ and $\xi_{2\ell} = \xi'$, such that (5.10) holds. This yields

$$(\xi_{2i}, \xi_{2i+2}) \in \boldsymbol{\psi}^\top, \quad i = 0, \cdots, \ell - 1,$$

which implies that $(\xi, \xi') \in \boldsymbol{\psi}^{\top+=}$. This provides the opposite inclusion $\boldsymbol{\psi}^{+=} \cap (\mathfrak{X}^\top \times \mathfrak{X}^\top) \subset \boldsymbol{\psi}^{\top+=}$ and thus $\boldsymbol{\psi}^{+=} \cap (\mathfrak{X}^\top \times \mathfrak{X}^\top) = \boldsymbol{\psi}^{\top+=}$.

Since $\leqslant$ restricted to $\mathfrak{X}^\top$ is an anti-chain the restriction of $\leqslant^\dagger$ to $\mathfrak{X}^\top$ is equal to the restriction of $\leqslant^+$ to $\mathfrak{X}^\top$, i.e. $\boldsymbol{\psi}^{+=} \cap (\mathfrak{X}^\top \times \mathfrak{X}^\top)$. Therefore, the restriction of $\leqslant^\dagger$ to $\mathfrak{X}^\top$ is equal to $\boldsymbol{\psi}^{\top+=} \cap (\mathfrak{X}^\top \times \mathfrak{X}^\top)$, which completes the proof. □

By Theorem 5.23 the transitive, reflexive closure $\boldsymbol{\psi}^{\top+=}$ is a closure operator on $\mathfrak{X}^\top$, i.e. $\mathbf{cl}^\top = (\boldsymbol{\psi}^{\top+=})^{-1}$, and is thus a condensed Morse pre-order for φ. By Theorem 5.22 the braid class components correspond to the partial equivalence classes of $\boldsymbol{\psi}^{+=}$ and therefore with the partial equivalence classes of $\boldsymbol{\psi}^{\top+=}$. By the

[7] Recall that $G_{d-1}\mathfrak{X}$ is the skeleton of co-dimension one cells in $\mathfrak{X}$.

order-preserving map $\mathrm{dyn}\colon (\mathfrak{X}, \leqslant) \twoheadrightarrow (\mathsf{SC}, \leqslant)$ given in (3.18) we obtain the partial equivalence classes of $\leqslant^{\dagger}$. The interior yields the braid class components: the open sets $\mathrm{int}\,|\mathrm{dyn}^{-1}\mathcal{S}|$, $\mathcal{S} \in \mathsf{SC}$, describe all discrete braid class components. The advantage of $|\mathrm{dyn}^{-1}\mathcal{S}|$ is that its Borel-Moore homology gives its Conley index via $H^{\mathrm{tile}}(\mathcal{S})$. Another way to retrieve the braid class components from $\leqslant^{\top}$ is to use Section 3.4.2. In this way we obtain the closures of the braid class components. This way one cannot immediately determine its Conley index

In the next section we carry out a specific analysis for a number of examples of parabolic systems.

5.4. Recipe for global decompositions

In this section we apply the methodology of this text to parabolic flows in combination with the theory of discretized braids. This application will use all of the ingredients described in the previous chapters. We also place an emphasis on the computational aspects and highlight how these can be carried out in practice. The goal is to obtain a Morse pre-order for a discrete parabolic flow φ, which encodes the directionality of φ, from which we can determine a (graded) tessellar differential module, a graded representation (connection matrix), and a tessellar phase diagram, whose structures reveal information about the invariance and connecting orbits of φ.

Section 5.4.1 outlines the general recipe for computing a Morse pre-order and a graded representation applied to parabolic flows. In order to make use of the algorithm ConnectionMatrix of [**31**], we also assume that we work with homology over fields. Section 5.4.2 is specific to parabolic flows, and introduces parabolic Betti numbers/homology using the lap number grading.

5.4.1. Computing Morse pre-orders and tessellar chain complexes. We divide the computations into three appropriate steps: topologization, discretization and algebraization. These steps use the tools of graph theory and computational algebraic topology.[8]

5.4.1.1. *Topologization.* (a) The space of d-periodic sequences is a cube in $\mathbb{R}^d$ and is given the standard metric topology. The block-flow topology given by a parabolic flow is derived from the backward image operator as explained in Section 5.3. The idea in Section 3 is to construct a pre-order that discretizes both the metric topology as well as the block-flow topology. This is carried out such that CW-discretization map $\mathrm{cell}\colon X \twoheadrightarrow \mathfrak{X}$ has the right continuity properties.

[8]These computations can be set up for a given skeleton using the open-source software package PyChomp [**33**]. Of note is that the software is very efficient, and can calculate condensed Morse pre-orders and connection matrices for examples of parabolic flows with $|\mathfrak{X}| \approx 2.5 \times 10^{10}$, and $|\mathsf{SC}| \approx 6.2 \times 10^4$, cf. [**32**]. More details on the software and algorithms, in addition to timing information for computational experiments, can be found in [**31**, **32**].

5.4.1.2. *Discretization.* We breakdown discretization into steps (b)–(f); steps (b)–(d) are represented in Fig.'s 5.1–5.2 and step (e) in Fig. 5.3.

(b) For parabolic flows we use specific discretizations that are compatible with the braid classes for a given skeleton y, i.e., the top-cells $\mathfrak{X}^\top$ correspond to generic braids given by $G_d X$. For a given skeletal braid $y \in \mathscr{D}^d_m$ that is stationary for φ and for which $\mathring{y}$ is proper, the phase space is given by Equation (5.4). Following the representation in Remark 5.11 we represent y in normal form[9] which yields a cubical CW-decomposition $\mathfrak{X}$ with the appropriate number of cubes, q.v. Fig. 5.3.

(c) The top cells $\xi \in \mathfrak{X}^\top$ of the cubical CW-decomposition described in (a) correspond to the subsets $|\xi| \subset G_d X$ of generic braids $x \text{ rel } y \in \mathscr{D}^d_1 \text{ rel } y$, cf. Sect. 5.3.1. For the top cells we determine the symmetric adjacency relation $\boldsymbol{\mathcal{C}}^\top \subset \mathfrak{X}^\top \times \mathfrak{X}^\top$ as described in Section 5.3.2.

(d) For every generic braid diagram $x \text{ rel } y$ the crossing number $\boldsymbol{\lambda}(x \text{ rel } y) = \boldsymbol{\lambda}(\xi) \in \mathbb{N}$ is well-defined and can be given as the crossing number $\boldsymbol{\lambda}(\xi)$ of the unique top cell representating $x \text{ rel } y$. From the description in Section 5.3.2 we obtain the generating relation $\boldsymbol{\psi}^\top \subset \boldsymbol{\mathcal{C}}^\top$ for the condensed Morse pre-order $\leqslant^\top$ via $(\xi, \xi') \in \boldsymbol{\psi}^\top$ if and only if $\boldsymbol{\lambda}(\xi) \leqslant \boldsymbol{\lambda}(\xi')$.

(e) For the relation $\boldsymbol{\psi}^\top$ we compute the poset of strongly connected components $(\mathsf{SC}, \leqslant)$. This can be done in time $O(|\mathfrak{X}^\top| + |\boldsymbol{\psi}^\top|)$ using Tarjan's algorithm [64].[10] The elements $\mathcal{S} \in \mathsf{SC}$ correspond to discrete braid class components via $\operatorname{int} |\operatorname{cl} \mathcal{S}|$, where cl is closure in $(\mathfrak{X}, \leqslant)$ and int is interior in $(X, \mathscr{T})$.

(f) We use the formula for the map $\operatorname{dyn}\colon \mathfrak{X} \twoheadrightarrow \mathsf{SC}$ given by Theorem 3.29 to reconstruct the pre-order $(\mathfrak{X}, \leqslant^\dagger)$. The partial equivalence classes of $\leqslant^\dagger$ are given by $\operatorname{dyn}^{-1}\mathcal{S}$. Note that $\operatorname{cl} \operatorname{dyn}^{-1}\mathcal{S} = \operatorname{cl} \mathcal{S}$. The difference between $\operatorname{dyn}^{-1}\mathcal{S}$ and $\operatorname{cl} \mathcal{S}$ is that the former is convex in $(\mathfrak{X}, \leqslant^\dagger)$ and thus locally closed in $(\mathfrak{X}, \leqslant)$. This implies that $|\operatorname{dyn}^{-1}\mathcal{S}|$ is locally compact, and that the Borel-Moore homology is well-defined and can be computed via the cellular homology. The pre-order $(\mathfrak{X}, \leqslant^\dagger)$ defines a Morse tessellation. The Morse tiles are given by the formula $|\operatorname{dyn}^{-1}\mathcal{S}| = G_{\mathcal{S}} X$, cf. Eqn. (4.23). Having the pre-order $(\mathfrak{X}, \leqslant^\dagger)$ now establishes the discretization map

$$\begin{tikzcd} X \arrow[r, two heads, "\mathrm{cell}"] \arrow[rr, two heads, bend right, "\mathrm{tile}"'] & \mathfrak{X} \arrow[r, two heads, "\mathrm{dyn}"] & \mathsf{SC} \end{tikzcd}$$

which also induces a non-trivial grading of X.

5.4.1.3. *Algebraization.* The steps (a)–(f) yield the cubical CW-decomposition $\mathfrak{X}$, the poset $(\mathsf{SC}, \leqslant)$ and the map $\operatorname{dyn}\colon \mathfrak{X} \twoheadrightarrow \mathsf{SC}$.

(g) The CW-decomposition map $\operatorname{cell}\colon (X, \mathscr{T}) \twoheadrightarrow (\mathfrak{X}, \leqslant)$ together with the map $\operatorname{dyn}\colon (\mathfrak{X}, \leqslant) \twoheadrightarrow (\mathsf{SC}, \leqslant)$ form an SC-graded cell complex, cf. [31], which is the input for the algorithm CONNECTIONMATRIX of [31].

[9]None of the results here are affected by choosing a normal form because all skeleta are homotopic, cf. [24].

[10]Note that without a condensed Morse pre-order it would take time $O(|\mathfrak{X}| + |\leqslant^\dagger|)$ to compute the poset SC.

$$0 \to \bigoplus_{i\in\{6,8,10\}} \mathbb{Z}_2\langle \mathcal{S}_i\rangle \xrightarrow{\mathrm{d}_1} \bigoplus_{i\in\{0,1,3,4\}} \mathbb{Z}_2\langle \mathcal{S}_i\rangle \to 0, \quad \mathrm{d}_1 = \begin{array}{c c} & \begin{array}{ccc} \mathcal{S}_6 & \mathcal{S}_8 & \mathcal{S}_{10} \end{array} \\ \begin{array}{c} \mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_3 \\ \mathcal{S}_4 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

FIGURE 5.4. Graded tessellar differential module $C^{\mathrm{tile}}(X)$ for the example in Figure 5.1[left]. The differential $\mathrm{d}^{\mathrm{tile}}$ is computed using $\mathbb{Z}_2$ coefficients [right].

As output, we obtain the graded tessellar differential module $C^{\mathrm{tile}}(X)$, as described in Section 4.4, cf. Fig. 5.4. In particular, we obtain the Borel-Moore homologies $H^{\mathrm{BM}}(G_{\mathcal{S}}X)$, and all Borel-Moore homologies are finitely generated. In general the time complexity of this step is $O(|\mathfrak{X}|^3)$, however in practice it is linear [32].

(h) Since the homology is computed over a field, i.e. $\mathbb{K} = \mathbb{Z}_2$, it is completely described by its Betti numbers/Poincaré polynomials. We visualize the ensemble of Morse pre-order and tessellar differential module by augmenting the Hasse diagram for SC by providing the Borel-Moore Poincaré polynomials of the Morse tiles $G_{\mathcal{S}}X$. This visualization of the tessellar phase diagram $(\amalg, \leqslant^{\dagger})$ in given in Figure 5.5, cf. Sect. 4.4.

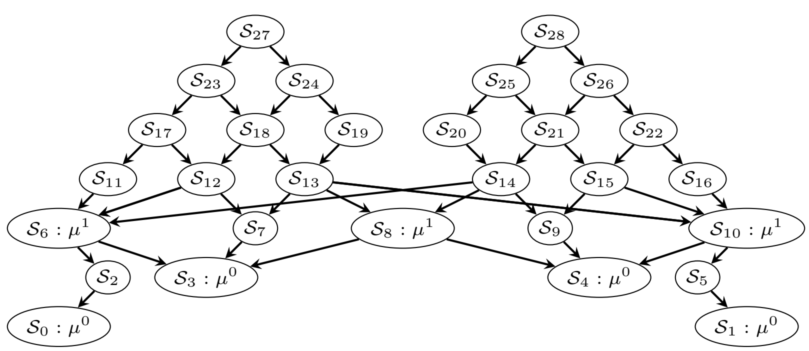

FIGURE 5.5. Tessellar phase diagram for $(\amalg, \leqslant^{\dagger})$ for the example in Figure 5.1. The elements in $(\mathsf{SC}, \leqslant)$ are labeled with the natural numbers. Elements $\mathcal{S} \in \mathsf{SC}$ with trivial tessellar homology are indicated only by label.

This structure is an algebraic and combinatorial description of the global dynamics of φ, encoding both the directionality and the invariance. We can also display the tessellar differential (connection matrix data), although in practice we regard this as a separate, queryable data structure which lives over the tessellar phase diagram. The pure tessellar phase diagram is obtained by the subposet of vertices with non-trivial homology, cf. Fig. 5.6.

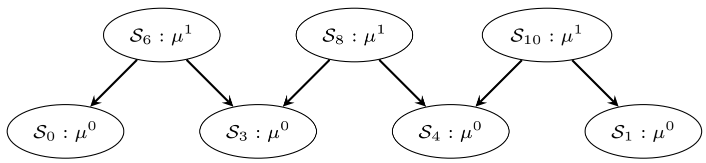

FIGURE 5.6. The pure tessellar phase diagram $(\overline{\Pi}, \leqslant^{\ddagger})$ for the example in Figure 5.1.

5.4.2. Lap number grading and parabolic homology. From this point on we use field coefficients $R = \mathbb{K}$. In the above calculations we compute the tessellar homology associated with the discretization $\mathrm{tile}\colon X \to \mathsf{SC}$. The elements of SC represent discretized braid class components and therefore the crossing number $\boldsymbol{\lambda}(\mathcal{S}) \in 2\mathbb{N}$ is well-defined for all $\mathcal{S} \in \mathsf{SC}$, i.e. we have an order-preserving map $\mathcal{S} \mapsto \frac{1}{2}\boldsymbol{\lambda}(\mathcal{S})$. The latter may be regarded as a discretization map and will be denoted by $\mathrm{lap}\colon \mathsf{SC} \to \mathbb{N}$. This yields the following factorization

$$
\begin{tikzcd}
 & \mathsf{SC} \arrow[d, "\mathrm{lap}"] \\
X \arrow[ur, "\mathrm{tile}"] \arrow[r, "\mathrm{pb}"'] & \mathbb{N}
\end{tikzcd}
\tag{5.11}
$$

where $\mathrm{pb}\colon X \to \mathbb{N}$ is the composition of tile and lap. By the same token as before the discretization lap yields an $\mathbb{N}$-graded differential module/chain complex $\big(C^{\mathrm{tile}}, \mathrm{d}^{\mathrm{tile}}\big)$ and an $\mathbb{N}$-grading of the tessellar homology H^{tile}. Following the procedure in Section 4.2.3 we have:

DEFINITION 5.24. The scalar discretization $\mathrm{lap}\colon \mathsf{SC} \to \mathbb{N}$ yields a bi-graded homology theory for H^{tile} which is be denoted by

$$
\vec{H}_{p,q}(X) := G_p H_q^{\mathrm{tile}}(X), \quad p, q \in \mathbb{N}, \tag{5.12}
$$

and will be referred to as the *parabolic homology*.[11]

The parabolic homology is defined for all sets $G_{\mathcal{U} \smallsetminus \mathcal{U}'} X$, with $\mathcal{U} \smallsetminus \mathcal{U}'$ convex in $(\mathsf{SC}, \leqslant)$. In particular, if we restrict to the convex sets $\{\mathcal{S}\}$ in $(\mathsf{SC}, \leqslant)$ we obtain

$$
\vec{H}_{p,q}(G_{\mathcal{S}} X) = \begin{cases} H_q^{\mathrm{BM}}(G_{\mathcal{S}} X) & \text{for } p = \mathrm{lap}\, G_{\mathcal{S}} X; \\ 0 & \text{otherwise.} \end{cases} \tag{5.13}
$$

[11]The definition of the parabolic homology and the bi-grading uses the fact that we use field coefficients. In general we only obtain bi-graded Betti numbers.

For convex sets $\mathcal{U} \smallsetminus \mathcal{U}' \subset \mathfrak{X}$ we have:

$$\vec{P}_{\lambda,\mu}(G_{\mathcal{U}\smallsetminus\mathcal{U}'}X) = \sum_{p,q\in\mathbb{N}} \big(\operatorname{rank} \vec{H}_{p,q}(G_{\mathcal{U}\smallsetminus\mathcal{U}'}X)\big)\lambda^p\mu^q$$

and $P_{\lambda,\mu}\big(C^{\mathrm{tess}}(G_{\mathcal{U}\smallsetminus\mathcal{U}'}X)\big) = \sum_{\mathcal{S}\in\mathcal{U}\smallsetminus\mathcal{U}'}\sum_{p,q\in\mathbb{N}}\big(\operatorname{rank}\vec{H}_{p,q}(G_{\mathcal{S}}X)\big)\lambda^p\mu^q$. The Morse relation from Theorem 4.18 yield

$$\sum_{\mathcal{S}\in\mathsf{SC}} \vec{P}_{\lambda,\mu}(G_{\mathcal{S}}X) = \vec{P}_{\lambda,\mu}(X) + \sum_{r=1}^{\infty}(1+\lambda^r\mu)Q^r_{\lambda,\mu}, \tag{5.14}$$

with $Q^r_{\lambda,\mu} \geqslant 0$.

The matrices below describe the tessellar differential with q and p grading respectively for the skeleton y in Figure 5.1.

$$\begin{array}{c} \\ \mathcal{S}_0\\ \mathcal{S}_1\\ \mathcal{S}_3\\ \mathcal{S}_4\\ \mathcal{S}_6\\ \mathcal{S}_8\\ \mathcal{S}_{10}\end{array}
\begin{array}{c}\begin{array}{cccc|ccc}\mathcal{S}_0 & \mathcal{S}_1 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_6 & \mathcal{S}_8 & \mathcal{S}_{10}\end{array}\\
\left(\begin{array}{cccc|ccc}
 & & & & 1 & 0 & 0\\
 & & & & 0 & 0 & 1\\
 & & & & 1 & 1 & 0\\
 & & & & 0 & 1 & 1\\ \hline
 & & & & & & \\
 & & & & & & \\
 & & & & & &
\end{array}\right)\end{array}
\qquad
\begin{array}{c} \\ \mathcal{S}_0\\ \mathcal{S}_1\\ \mathcal{S}_3\\ \mathcal{S}_4\\ \mathcal{S}_6\\ \mathcal{S}_8\\ \mathcal{S}_{10}\end{array}
\begin{array}{c}\begin{array}{cc|cc|ccc}\mathcal{S}_0 & \mathcal{S}_1 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_6 & \mathcal{S}_8 & \mathcal{S}_{10}\end{array}\\
\left(\begin{array}{cc|cc|ccc}
 & & 0 & 0 & 1 & 0 & 0\\
 & & 0 & 0 & 0 & 0 & 1\\ \hline
 & & & & 1 & 1 & 0\\
 & & & & 0 & 1 & 1\\ \hline
 & & & & & & \\
 & & & & & & \\
 & & & & & &
\end{array}\right)\end{array}$$

Using $\mathbb{K} = \mathbb{Z}_2$ the left matrix is $\mathrm{d}^{\mathrm{tile}}$ as chain complex boundary for $C^{\mathrm{tess}}(X) = \bigoplus_{q\in\{0,1\}} C^{\mathrm{tile}}_q(X)$, with

$$C^{\mathrm{tile}}_0(X) = \bigoplus_{i\in\{0,1,3,4\}} \mathbb{Z}_2\langle\mathcal{S}_i\rangle, \quad \text{and} \quad C^{\mathrm{tess}}_1(X) = \bigoplus_{i\in\{6,8,10\}} \mathbb{Z}_2\langle\mathcal{S}_i\rangle.$$

The right matrix is $\mathrm{d}^{\mathrm{tile}}$ as differential for the $\mathbb{Z}$-graded vector space $C^{\mathrm{tile}}(X) = \bigoplus_{p\in\{0,1,2\}} G_pC^{\mathrm{tile}}(X)$, with

$$G_0C^{\mathrm{tile}}(X) = \bigoplus_{i\in\{0,1\}} \mathbb{Z}_2\langle\mathcal{S}_i\rangle,\ G_1C^{\mathrm{tile}}(X) = \bigoplus_{i\in\{3,4\}} \mathbb{Z}_2\langle\mathcal{S}_i\rangle,\ G_2C^{\mathrm{tile}}(X) = \bigoplus_{i\in\{6,8,10\}} \mathbb{Z}_2\langle\mathcal{S}_i\rangle.$$

Figure 5.4 depicts $C^{\mathrm{tile}}(X)$ as chain complex and Figure 5.7 below depicts $C^{\mathrm{tile}}(X)$ the differential vector space. If we determine the parabolic homology of the sets

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i\in\{6,8,10\}} \mathbb{Z}_2\langle\mathcal{S}_i\rangle \arrow[r, "\left(\begin{smallmatrix}1&1&0\\0&1&1\end{smallmatrix}\right)"] \arrow[rr, bend right, "\left(\begin{smallmatrix}1&0&0\\0&0&1\end{smallmatrix}\right)"'] & \bigoplus_{i\in\{3,4\}} \mathbb{Z}_2\langle\mathcal{S}_i\rangle \arrow[r, "\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)"] & \bigoplus_{i\in\{0,1\}} \mathbb{Z}_2\langle\mathcal{S}_i\rangle \arrow[r] & 0
\end{tikzcd}$$

FIGURE 5.7. The tessellar differential module $C^{\mathrm{tile}}(X)$ with the terms of the tessellar differential acting on the different groups $G_pC^{\mathrm{tile}}(X)$, $p = 0, 1, 2$.

$\mathcal{S} \in \mathsf{SC}$ in Figure 5.1 we obtain a refinement of the reduced tessellar phase given in Figure 5.8. To compute the bi-graded homology we use the spectral recurrence procedure. To illustrate the lap number grading we start with computing $\vec{H}_{p,q}(X)$

for all p, q. Recall that G_pX is a convex set and is the union of braid classes with lap number p and $F_{\downarrow p}X$ is the union of braid classes with lap number less or equal to p. We compute the homologies from the chain complex $(C^{\mathrm{tile}}, \mathrm{d}^{\mathrm{tile}})$. We have the following short exact sequences:

$$
\begin{aligned}
0 \to F_{\downarrow 0}C_0^{\mathrm{tile}} &\xrightarrow{i_{0,0}} F_{\downarrow 1}C_0^{\mathrm{tile}} \xrightarrow{j_{1,0}} G_1C_0^{\mathrm{tile}} \to 0;\\
0 \to F_{\downarrow 1}C_0^{\mathrm{tile}} &\xrightarrow[\cong]{i_{1,0}} F_{\downarrow 2}C_0^{\mathrm{tile}} \xrightarrow{j_{2,0}} G_2C_0^{\mathrm{tile}} \to 0;\\
0 \to F_{\downarrow 1}C_1^{\mathrm{tile}} &\xrightarrow{i_{1,1}} F_{\downarrow 2}C_1^{\mathrm{tile}} \xrightarrow{j_{2,1}} G_2C_1^{\mathrm{tile}} \to 0,
\end{aligned}
$$

and the isomorphisms

$$F_{\downarrow 0}C_0^{\mathrm{tile}} \xrightarrow[\cong]{j_{0,0}} G_0C_0^{\mathrm{tile}}, \quad F_{\downarrow 1}C_1^{\mathrm{tile}} \xrightarrow[\cong]{j_{1,1}} G_1C_1^{\mathrm{tile}} \cong 0, \quad G_2C_0^{\mathrm{tile}} \cong 0.$$

The vector spaces are generated by the non-trivial classes given in Figure 5.4. To build the spectral sequence we have have page 1: $E^1_{0,0} = H_0^{\mathrm{tile}}(G_0X) \cong \mathbb{Z}_2^2$, $E^1_{1,0} = H_0^{\mathrm{tile}}(G_1X) \cong \mathbb{Z}_2^2$ and $E^1_{2,1} = H_1^{\mathrm{tile}}(G_2X) \cong \mathbb{Z}_2^3$, cf. Sect. 4.2.3. Furthermore, $E^1_{0,1}$, $E^1_{1,0}$, $E^1_{1,1}$, $E^1_{2,2}$ and $E^1_{p,q}$, for $p, q > 2$, are trivial. The relevant spectral sequences are:

$$0 \to E^1_{2,1} \xrightarrow{\mathrm{d}^1_{2,1}} E^1_{1,0} \to 0, \quad 0 \to E^1_{2,1} \xrightarrow{\mathrm{d}^1_{1,1}} E^1_{1,0} \to 0,$$

where $E^1_{1,1} = 0$ and $\mathrm{d}^1_{1,1} = 0$. The differential $\mathrm{d}^1_{2,1}$ can be determined from the above data. From the the third short exact sequence above we have

$$\cdots \to H_1^{\mathrm{tile}}(F_{\downarrow 1}X) \xrightarrow{i_{1,1}} H_1^{\mathrm{tile}}(F_{\downarrow 2}X) \xrightarrow{j_{2,1}} H_1^{\mathrm{tile}}(G_2X) \xrightarrow{k_{2,1}} H_0^{\mathrm{tile}}(F_{\downarrow 1}X) \to \cdots,$$

where $k_{2,1}$ is the connecting homomorphism. The vector space $E^1_{2,1}$ is given by $\mathrm{d}_1(\mathcal{U})\colon G_2C_1^{\mathrm{tile}} \to G_1C_0^{\mathrm{tile}}$, where $\mathrm{d}_1(\mathcal{U})$, with $\mathcal{U} = \mathrm{lap}^{-1}2$, is the zero matrix. Let $\gamma = (a, b, c) \in \bigoplus_{i\in\{6,8,10\}} \mathbb{Z}_2\langle \mathcal{S}_i\rangle$ be a cycle. Then, the inverse image under $j_{2,1}$ is the same element $\gamma \in F_{\downarrow 2}C_1^{\mathrm{tile}}$. Apply $\mathrm{d}^{\mathrm{tile}}$, i.e. $\mathrm{d}_1\gamma = (a, c, a+b, b+c) \in F_{\downarrow 1}C_0^{\mathrm{tile}}$, which is also the homology class in $H_0^{\mathrm{tile}}(F_{\downarrow 1}X)$. This calculation yields: $k_{2,1} = \mathrm{d}_1$. From the the first short exact sequence above we have

$$0 \to H_0^{\mathrm{tile}}(F_{\downarrow 0}X) \xrightarrow{i_{0,0}} H_0^{\mathrm{tile}}(F_{\downarrow 1}X) \xrightarrow{j_{1,0}} H_0^{\mathrm{tile}}(G_1X) \to 0,$$

where the map $j_{1,0}$ is given by $(a, b, c, d) \mapsto (0, 0, c, d)$. The differential $\mathrm{d}^1_{2,1}$ is given by $\mathrm{d}^1_{2,1} = j_{1,0} \cdot k_{2,1}$ and

$$\mathrm{d}^1_{2,1} = \begin{pmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix} \begin{pmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 1\,1\,0 \\ 0\,1\,1 \end{pmatrix} = \begin{pmatrix} 1\,1\,0 \\ 0\,1\,1 \end{pmatrix} : \mathbb{Z}_2^3 \longrightarrow \mathbb{Z}_2^2.$$

The next page of the spectral sequence yields the vector spaces

$$E^2_{2,1} = \ker \mathrm{d}^1_{2,1} \cong \mathbb{Z}_2,\ E^2_{1,0} = \frac{\mathbb{Z}_2^2}{\mathrm{im}\ \mathrm{d}^1_{2,1}} \cong 0,\ E^2_{1,1} = 0,\ E^2_{0,0} \cong \frac{\mathbb{Z}_2^2}{\mathrm{im}\ \mathrm{d}^1_{1,1}} \cong \mathbb{Z}_2^2.$$

To complete the lap number grading of tessellar homology we compute the third page of the spectral sequence. The remaining spectral sequence is

$$0 \to E^2_{2,1} \xrightarrow{\mathrm{d}^2_{2,1}} E^2_{0,0} \to 0,$$

where the differential $\mathrm{d}^2_{2,1}$ is computed as follows. From the theory of spectral sequences we have that $\mathrm{d}^2_{2,1} = j_{0,0} \cdot i^{-1}_{0,0} \cdot k_{2,1}$. Let $\gamma = (a,a,a) \in E^2_{2,1}$, then $k_{2,1}\gamma = (a,a,0,0) \in E^1_{1,0} = H^{\mathrm{tile}}_0(F_1 X)$. Under $i^{-1}_{0,0}$ we obtain $i^{-1}_{0,0}(a,a,0,0) = (a,a) \in H^{\mathrm{tile}}_0(F_{\downarrow 0}\mathfrak{X})$. Since $F_{\downarrow 0}C^{\mathrm{tile}}_0 \xrightarrow[=]{j_{0,0}} G_0 C^{\mathrm{tile}}_0$ the map $j_{0,0}$ is the identity which implies that $\mathrm{d}^2_{2,1}$ is given by $(a,a,a) \mapsto (a,a) \in E^2_{0,0}$. This map is the restriction to $E^2_{2,1}$ of the map

$$\widetilde{\mathrm{d}}^2_{2,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : E^1_{2,1} \longrightarrow E^1_{0,0}.$$

Consequently, $\ker \mathrm{d}^2_{2,1} = 0$ and thus $E^3_{2,1} = 0$. Moreover, $\mathrm{im}\, \mathrm{d}^2_{2,1} \cong \mathbb{Z}_2$ and thus $E^3_{0,0} = \frac{\mathbb{Z}^2_2}{\mathrm{im}\, \mathrm{d}^2_{2,1}} \cong \mathbb{Z}_2$. The spectral sequence stabilizes at $r \geqslant 3$ and we define $\vec{H}_{p,q}(X) := E^r_{p,q}$, $r \geqslant 3$. In particular,

$$\vec{H}_{p,q}(X) = \begin{cases} \mathbb{Z}_2 & \text{for } (p,q) = (0,0); \\ 0 & \text{for } (p,q) \neq (0,0), \end{cases}$$

and thus $\vec{P}_{\lambda,\mu}(X) = 1$. The same procedure can be carried out for other convex sets in SC. If we apply the Morse relations in (5.14) for the skeleton y in Figure 5.1

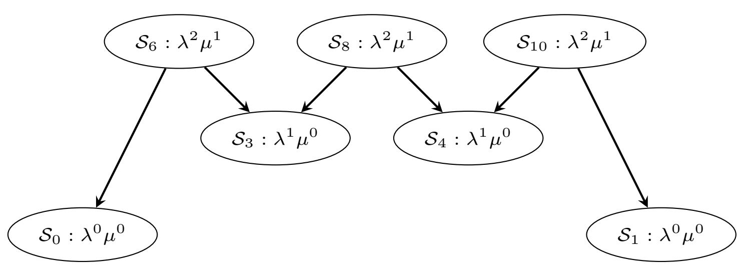

FIGURE 5.8. The pure tessellar phase diagram $(\overline{\Pi}, \leqslant^{\ddagger})$ with parabolic Poincaré polynomials. The vertices are positions with height corresponding to their lap numbers.

we obtain

$$P_{\lambda,\mu}(C^{\mathrm{tile}}) = 2 + 2\lambda + 3\lambda^2\mu = 1 + (1+\lambda\mu)\cdot 2\lambda + (1+\lambda^2\mu)\cdot 1,$$

which implies that $Q^1_{\lambda,\mu} = 2\lambda$ and $Q^2_{\lambda,\mu} = 1$. This provides information about the differentials $\mathrm{d}^1_{2,1}$ and $\mathrm{d}^2_{2,1}$ in the associated spectral sequence. Note that the sum of the ranks of $\mathrm{d}^1_{2,1}$ and $\mathrm{d}^2_{2,1}$ equals the rank of $\mathrm{d}^{\mathrm{tile}}$.

REMARK 5.25. If we apply the Cartan-Eilenberg theory, and Theorem 4.7 in particular, to the discretization $\mathrm{pb}\colon X \to \mathbb{Z}$ given by the composition $X \xrightarrow{\text{tile}} \mathsf{SC} \xrightarrow{\text{lap}} \mathbb{Z}$ we obtain tessellar differential of the form:

$$\mathrm{d}^{\mathrm{pb}} = \begin{array}{c c} & \begin{array}{ccc} E^1_{0,0} & E^1_{1,0} & E^1_{2,1} \end{array} \\ \begin{array}{c} E^1_{0,0} \\ E^1_{1,0} \\ E^1_{2,1} \end{array} & \begin{pmatrix} 0 & 0 & \widetilde{\mathrm{d}}^2_{2,1} \\ 0 & 0 & \mathrm{d}^1_{2,1} \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

This is exactly the tessellar boundary operator $\mathrm{d}^{\mathrm{tile}}$ as given in Fig. 5.4. The entries of d^{pb} can also be marked as $\mathrm{d}^r_{p,q}$. In this case $\mathrm{d}^1_{1,0} = 0$, $\mathrm{d}^1_{2,1} = \mathrm{d}^1_{2,1}$ and $\mathrm{d}^2_{2,1} = \widetilde{\mathrm{d}}^2_{2,1}$.

5.5. Differential modules and tessellar phase diagrams for positive braids

For a parabolic flow φ with a proper skeleton y, i.e. $\mathring{y}$ is proper, we obtained a canonical Morse pre-order $\leqslant^\dagger$ via a CW-decomposition induced by y and the parabolic recurrence relation $\mathbf{R}$. The poset SC induced by y describes the Morse tessellation of all discrete braid class components $[x] \text{ rel } y$. The Cartan-Eilenberg theory then provides an algebraic topological data structure that contains topological information about the braid class components (Morse tiles) and algebraic information on how the classes are stitched together. We give a substantial extension of the results in [24] by showing that these data structures are invariants of positive conjugacy classes of braids.

5.5.1. Some braid theory. Following [24, 65, 14] we recall some basic ideas from braid theory. Let $x \in \mathscr{D}^d_n$ be a discrete braid diagram. Generically strands in x intersect with $x^\alpha_i \neq x^{\alpha'}_i$, cf. Rmk. 5.3. To such generic braid diagrams $x \in \mathscr{D}^d_n$ one assigns a unique positive word $\beta = \beta(x)$ given by:

$$x \mapsto \beta(x) = \sigma_{\alpha_1} \cdots \sigma_{\alpha_\ell}, \tag{5.15}$$

where α_k and $\alpha_k + 1$ are the positions of intersection that intersect, cf. Rmk. 5.11. The (algebraic) *Artin braid group* $\mathscr{B}_n$ is a free group spanned by the $m-1$ generators σ_α, modulo following relations:

$$\begin{cases} \sigma_\alpha \sigma_{\alpha'} = \sigma_{\alpha'} \sigma_\alpha, & |\alpha - \alpha'| \geqslant 2,\ \alpha, \alpha \in \{0, \ldots, n-2\} \\ \sigma_\alpha \sigma_{\alpha+1} \sigma_\alpha = \sigma_{\alpha+1} \sigma_\alpha \sigma_{\alpha+1}, & 0 \leqslant \alpha \leqslant n-3. \end{cases} \tag{5.16}$$

Presentations of words consisting only of the σ_i's (not the inverses) and the relations in (5.16) form a monoid which is called the *positive braid monoid* $\mathscr{B}^+_n$. Two positive words β and β' are *positively equal* if they represent the same element in $\mathscr{B}^+_n$ by using the σ-relations in the braid group. Notation $\beta \stackrel{+}{=} \beta'$. Two positive words β, β' are *positively conjugate* if there exists a sequence of words $\beta_0, \cdots, \beta_\ell \in \mathscr{B}^+_n$, with $\beta_0 = \beta$ and $\beta_\ell = \beta'$, such that for all k, either $\beta_k \stackrel{+}{=} \beta_{k+1}$, or $\beta_k \equiv \beta_{k+1}$, where the latter is defined by

$$\sigma_{\alpha_1} \sigma_{\alpha_2} \cdots \sigma_{\alpha_p} \equiv \sigma_{\alpha_2} \cdots \sigma_{\alpha_p} \sigma_{\alpha_1},$$

cf. [65, Sect. 2.2]. Notation $\beta \overset{+}{\sim} \beta'$. Positive conjugacy is an equivalence relation on $\mathscr{B}_n^+$ and the positive conjugacy class of $\beta \in \mathscr{B}_n^+$ is denoted by $\langle\beta\rangle$. The set of all positive conjugacy classes in $\mathscr{B}_n^+$ is denoted by $\langle\mathscr{B}_n^+\rangle$.

DEFINITION 5.26. Two discretized braids $x, x' \in \mathscr{D}_n^d$ are *topologically equivalent* if $\beta(x)$ and $\beta(x')$ are positively conjugate. Notation: $x \overset{+}{\sim} x'$.

Recall that $x \sim x'$ if and only if $x, x' \in [x]$. Clearly, $x \sim x'$ implies $x \overset{+}{\sim} x'$ which defines a coarser equivalence relation on $\mathscr{D}_n^d$. Denote the equivalence classes with respect to $\overset{+}{\sim}$ by $[x]_{\overset{+}{\sim}}$. The converse is not true in general, cf. [24, Fig. 8]. Following [24, Def. 17], a discretized braid class $[x]$ is *free* if $[x] = [x]_{\overset{+}{\sim}}$.

REMARK 5.27. For non-generic $x \in \mathscr{D}_n^d$ we choose $\beta(x)$ to be any representative in the positive conjugacy class $\langle\beta(x')\rangle$, for any $x' \sim x$.

PROPOSITION 5.28 ([24], Prop. 27). *If $d > \boldsymbol{\lambda}(x)$, then $[x]$ is a free braid class.*

For the space of 2-colored discretized braid diagrams $\mathscr{D}_{1,m}^d$ there exists a natural embedding $\mathscr{D}_{1,m}^d \hookrightarrow \mathscr{D}_{1+m}^d$ by regarding $x \text{ rel } y$ as braid in $\mathscr{D}_{1+m}^d$. Via the embedding we define the notion of topological equivalence of two 2-colored discretized braids: $x \text{ rel } y \overset{+}{\sim} x' \text{ rel } y'$ if they are topologically equivalent as braids in $\mathscr{D}_{1+m}^d$. The associated equivalence classes are denoted by $[x \text{ rel } y]_{\overset{+}{\sim}}$, which are not necessarily connected sets in $\mathscr{D}_{1,m}^d$. A 2-colored discretized braid class $[x \text{ rel } y]$ is free if $[x \text{ rel } y] = [x \text{ rel } y]_{\overset{+}{\sim}}$. If $d > \boldsymbol{\lambda}(x \text{ rel } y)$, then $[x \text{ rel } y]$ is free by Proposition 5.28.

In Theorem 5.22 we showed that each partial equivalence class of $(\mathfrak{X}, \leqslant^\dagger)$ corresponds with braid class components $[x] \text{ rel } y$ of the fiber $\pi^{-1}(y)$. The Borel-Moore homology of $[x] \text{ rel } y$ is independent of the choice of parabolic recurrence relations $\mathbf{R}$ for which $\mathbf{R}(y) = 0$, cf. [24, Thm. 15(a)-(b)], and therefore the parabolic homology is independent of $\mathbf{R}$, i.e.

$$\vec{H}_{p,q}\big([x] \text{ rel } y; \varphi\big) = \vec{H}_{p,q}\big([x] \text{ rel } y; \varphi'\big). \tag{5.17}$$

A similar statement holds for the braid classes $[x \text{ rel } y]$. Let $x \text{ rel } y \sim x' \text{ rel } y'$ and let φ and φ' be parabolic flows with skeleton y and y' respectively. Then,

$$\bigoplus_{\substack{[x] \text{ rel } y \subset \\ \pi^{-1}(y) \cap [x \text{ rel } y]}} \vec{H}_{p,q}\big([x] \text{ rel } y; \varphi\big) \cong \bigoplus_{\substack{[x'] \text{ rel } y' \subset \\ \pi^{-1}(y') \cap [x' \text{ rel } y']}} \vec{H}_{p,q}\big([x'] \text{ rel } y'; \varphi'\big) \tag{5.18}$$

cf. [24, Thm. 15(c)]. This makes the latter an invariant of the discrete 2-colored braid class $[x \text{ rel } y]$ and justifies the notation:

$$\vec{H}_{p,q}\big([x \text{ rel } y]\big) := \bigoplus_{\substack{[x] \text{ rel } y \subset \\ \pi^{-1}(y) \cap [x \text{ rel } y]}} \vec{H}_{p,q}\big([x] \text{ rel } y; \varphi\big). \tag{5.19}$$

The homology $\vec{H}_{p,q}\big([x \text{ rel } y]\big)$ is not necessarily independent of to the number of discretization points d. In order to have independence also with respect to d, another invariant for discrete braid classes was introduced in [24]. Consider the equivalence class $[x \text{ rel } y]_{\overset{+}{\sim}}$ of discrete 2-colored braid diagrams induced by the relation $x \operatorname{rel} y \overset{+}{\sim} x' \operatorname{rel} y'$ on $\mathscr{D}^d_{1,m}$. As before the projection $\pi\colon \mathscr{D}^d_{1,m} \to \mathscr{D}^d_m$ given by $x \operatorname{rel} y \mapsto y$ yields the fibers $\pi^{-1}(y)$ and components $[x] \operatorname{rel} y \subset \pi^{-1}(y) \cap [x \operatorname{rel} y]_{\overset{+}{\sim}}$ and defines the homology:

$$\vec{H}_{p,q}\big([x \operatorname{rel} y]_{\overset{+}{\sim}}\big) := \bigoplus_{\substack{[x] \operatorname{rel} y \subset \\ \pi^{-1}(y) \cap [x \operatorname{rel} y]_{\overset{+}{\sim}}}} \vec{H}_{p,q}\big([x] \operatorname{rel} y; \varphi\big). \tag{5.20}$$

Note that if $d > \boldsymbol{\lambda}(x \operatorname{rel} y)$ then the homology in (5.20) corresponds to the homology in (5.19). As before $\vec{H}_{p,q}\big([x \operatorname{rel} y]_{\overset{+}{\sim}}\big)$ is an invariant and does not depend on the choice of φ and y. In the next section we explain the invariance by investigating the dependence on the number of discretization points d making it a topological invariant for relative braid classes.

The braid class components $[x] \operatorname{rel} y$ comprise the elements of SC and the equivalence relation $\overset{+}{\sim}$ yields a span which we express in terms of tessellar phase diagrams

$$\begin{tikzcd}
(\mathfrak{X}, \leqslant^\dagger) \arrow[r, two heads] \arrow[dr, two heads] & (\amalg, \leqslant^\dagger) \arrow[d, two heads] & \overline{\amalg}, \leqslant^\ddagger) \arrow[l, hook'] \arrow[d, two heads] \\
& (\amalg(y), \leqslant) & (\overline{\amalg}(y), \leqslant) \arrow[l, hook']
\end{tikzcd}$$

where $(\amalg(y), \leqslant)$ is the coarsening of $(\amalg, \leqslant^\dagger)$ defined by unionizing equivalent pairs $[x] \operatorname{rel} y, [x'] \operatorname{rel} y \subset \pi^{-1}(p) \cap [x \operatorname{rel} y]_{\overset{+}{\sim}}$ whenever $[x] \operatorname{rel} y \overset{+}{\sim} [x'] \operatorname{rel} y$ and taking transitive, reflexive closure. The equivalence classes are parallel in $\amalg$ and thus the coarsening yields a well-defined poset $(\amalg(y), \leqslant)$. The poset $(\overline{\amalg}(y), \leqslant)$ is the restriction of pairs in $\amalg(y)$ for which the Poincaré polynomials that are the non-zero. The elements of $\amalg(y)$ can be identified with $[x \operatorname{rel} y]_{\overset{+}{\sim}}$. By defining the groups in (5.19) we consider convex sets in SC. Therefore, the differential $\mathrm{d}^{\mathrm{tile}}$ yields a induced differential $\mathrm{d}^{\mathrm{para}}$ on the groups $\vec{H}_{p,q}\big([x \operatorname{rel} y]_{\overset{+}{\sim}}\big)$. This leads to the following definition:

DEFINITION 5.29. Let $y \in \mathscr{D}^d_m$ be a proper discrete braid. Define the differential module[12]

$$C^{\mathrm{para}}(X) := \bigoplus_{[x \operatorname{rel} y]_{\overset{+}{\sim}}} \bigoplus_{p,q} \vec{H}_{p,q}\big([x \operatorname{rel} y]_{\overset{+}{\sim}}\big), \qquad \mathrm{d}^{\mathrm{para}}\colon C^{\mathrm{para}}(X) \to C^{\mathrm{para}}(X). \tag{5.21}$$

where $\mathrm{d}^{\mathrm{para}}$ is the induced differential for $(\mathfrak{X}, \leqslant^\dagger)$. The non-trivial homologies $\vec{H}_{p,q}\big([x \operatorname{rel} y]_{\overset{+}{\sim}}\big) \neq 0$ yield an $\overline{\amalg}(y)$-grading of $C^{\mathrm{para}}(X)$. The $\overline{\amalg}(y)$-graded differential module $\mathscr{A} = \mathscr{A}(y) := \big(C^{\mathrm{para}}, \mathrm{d}^{\mathrm{para}}\big)$ is called the *parabolic differential module* for y.

Convex sets in $\overline{\amalg}(y)$ yield convex sets in $\amalg(y)$, i.e. $\mathsf{Co}(\overline{\amalg}(y)) \hookrightarrow \mathsf{Co}(\amalg(y))$[13] via convex hull: if $I \in \mathsf{Co}(\overline{\amalg}(y))$, then the convex hull $\langle I \rangle$ is contained in $\mathsf{Co}(\amalg(y))$. This implies that $C^{\mathrm{para}}(X)$ is well-defined over every convex set in $\mathsf{Co}(\overline{\amalg}(y))$. By

[12]We use the term differential module even though we use field coefficients which makes $\mathscr{A}$ a differential vector space.

[13]Recall that for a poset $(\mathsf{P}, \leqslant)$ the lattice of convex sets in P is denoted by $\mathsf{Co}(\mathsf{P})$, cf. [6].

construction the homology is given by $H(\mathscr{A}) \cong \mathbb{K}$. The grading by lap number is a first $\mathbb{Z}$-grading. The dimension grading of Borel-Moore homology comes as a second $\mathbb{Z}$-grading for $\mathscr{A}$. This yields the bi-graded homology $\vec{H}_{p,q}(\mathscr{A})$.

REMARK 5.30. For most of the examples in this paper the partial order on $\mathscr{A}$ is given by $\overline{\amalg}$, i.e. $\overline{\amalg} \cong \overline{\amalg}(y)$. In general this holds for d large enough.

REMARK 5.31. In general there does not exist an order-retraction $\amalg \twoheadrightarrow \overline{\amalg}$, cf. [40]. If such an order-retraction exists one obtains the discretization $(\mathfrak{X}, \leqslant^{\dagger}) \twoheadrightarrow (\amalg, \leqslant^{\dagger}) \twoheadrightarrow (\overline{\amalg}, \leqslant^{\ddagger})$ for which all tiles have non-trivial tessellar homology. It is not clear if this behaves well under stabilization. The same question applies to $\amalg(y)$ and $\overline{\amalg}(y)$. In the case of parabolic homology the remedy is two use the bi-grading and the order given by $\overline{\amalg}(y)$.

Figure 5.9 below shows a representation of $\mathscr{A}(y)$ for the discrete skeletal braid y displayed in Figure 1.5. The advantage of this representation is that the $\overline{\amalg}(y)$-grading is expressed in the diagram. Contracting to lap numbers yields the diagram in Figure 1.7.

$$
\begin{tikzcd}
 & & \mathbb{Z}_2\langle \mathcal{S}_8\rangle \arrow[dr, "(1)"] & & \mathbb{Z}_2\langle \mathcal{S}_5\rangle \arrow[dr, "(1)"] \arrow[drr, bend left, "\binom{0}{1}"] & & & \mathbb{Z}_2\langle \mathcal{S}_1\rangle \arrow[dr] & \\
0 \arrow[r] & \mathbb{Z}_2\langle \mathcal{S}_9\rangle \arrow[ur, "(1)"] \arrow[dr, "(1)"'] & & \mathbb{Z}_2\langle \mathcal{S}_6\rangle \arrow[ur, "(0)"] \arrow[dr, "(0)"'] & & \mathbb{Z}_2\langle \mathcal{S}_3\rangle \arrow[r, "\binom{0}{0}"] & \mathbb{Z}_2^2\langle \mathcal{S}_2\rangle \arrow[ur, "(1\,0)"] \arrow[dr, "(1\,0)"'] & & 0 \\
 & & \mathbb{Z}_2\langle \mathcal{S}_7\rangle \arrow[ur, "(1)"'] & & \mathbb{Z}_2\langle \mathcal{S}_4\rangle \arrow[ur, "(1)"'] & & & \mathbb{Z}_2\langle \mathcal{S}_0\rangle \arrow[ur] &
\end{tikzcd}
$$

FIGURE 5.9. The diagram displays the lap number grading as well as the $\overline{\amalg}$-grading of $\mathscr{A}(\beta)$. The $\mathbb{Z}_2$-groups are the parabolic homologies $\vec{H}_{p,q}(\mathcal{S}_j)$

5.5.2. Stabilization. In order to formulate the main results of this section we recall some definitions and methods from [24]. For a skeleton $y \in \mathscr{D}^d_m$ we define the extension operator $\mathbb{E}\colon \mathscr{D}^d_m \to \mathscr{D}^{d+1}_m$ as:

$$
(\mathbb{E}y)^{\alpha}_i := \begin{cases} y^{\alpha}_i & \text{for } i = 0, \cdots, d, \\ y^{\alpha}_d & \text{for } i = d+1. \end{cases}
$$

For a given braid class $[x \text{ rel } y]$ extension does not change any of the properties, i.e. $[\mathbb{E}x \text{ rel } \mathbb{E}y]$ is both bounded and non-degenerate. The same remains true under repeated application of the operator $\mathbb{E}$. The main result in [24, Thm. 20] states that the Borel-Moore homology of $[x \text{ rel } y]_{\stackrel{+}{\sim}}$ remains unchanged under application of $\mathbb{E}$. For the parabolic homology this implies:

THEOREM 5.32. *Let* $[x \text{ rel } y]_{\stackrel{+}{\sim}}$ *be a relative braid class then*

$$
\vec{H}_{p,q}\big([\mathbb{E}x \text{ rel } \mathbb{E}y]_{\stackrel{+}{\sim}}\big) \cong \vec{H}_{p,q}\big([x \text{ rel } y]_{\stackrel{+}{\sim}}\big) \tag{5.22}
$$

PROOF. The lap number does not change under the extension operator $\mathbb{E}$, i.e. $\mathrm{lap}\big([\mathbb{E}x \text{ rel } \mathbb{E}y]_{\dot\sim}\big) = \mathrm{lap}\big([x \text{ rel } y]_{\dot\sim}\big)$, where the latter is defined as the lap number of is braid class fibers. Since the Borel-Moore homologies are isomorphic by [24, Thm. 20] Equation (5.13) implies that the parabolic homologies are isomorphic. $\square$

By Proposition 5.28 $[x \text{ rel } y]$ a relative braid class is free if d is sufficiently large and thus $[\mathbb{E}^\ell x \text{ rel } \mathbb{E}^\ell y]$ is free for $\ell \geqslant 0$ sufficiently large. Let $x \text{ rel } y \,\dot\sim\, x' \text{ rel } y'$, then $\mathbb{E}^\ell x \text{ rel } \mathbb{E}^\ell y \,\dot\sim\, \mathbb{E}^\ell x' \text{ rel } \mathbb{E}^\ell y'$ and both are contained in the same connected component $[\mathbb{E}^\ell x \text{ rel } \mathbb{E}^\ell y]_{\dot\sim}$. By Equation (5.18) this implies that $\vec{H}_{p,q}\big([\mathbb{E}^\ell x \text{ rel } \mathbb{E}^\ell y]_{\dot\sim}\big) \cong \vec{H}_{p,q}\big([\mathbb{E}^\ell x' \text{ rel } \mathbb{E}^\ell y']_{\dot\sim}\big)$. Combining these isomorphisms gives

$$
\begin{aligned}
\vec{H}_{p,q}\big([x \text{ rel } y]_{\dot\sim}\big) &\cong \vec{H}_{p,q}\big([\mathbb{E}^\ell x \text{ rel } \mathbb{E}^\ell y]_{\dot\sim}\big) \cong \vec{H}_{p,q}\big([\mathbb{E}^\ell x' \text{ rel } \mathbb{E}^\ell y']_{\dot\sim}\big)\\
&\cong \vec{H}_{p,q}\big([x' \text{ rel } y']_{\dot\sim}\big),
\end{aligned}
$$

which establishes $\vec{H}_{p,q}\big([x \text{ rel } y]_{\dot\sim}\big)$ as a topological invariant for relative braid classes. The proof of [24, Thm. 20] is based on a singular perturbation argument for parabolic recurrence relations. We use the same technique now to show that the graded differential module $\mathscr{A}(y)$ is also invariant under extension by $\mathbb{E}$:

THEOREM 5.33. *Let $y \in \mathscr{D}^d_m$ be a proper discrete braid. Then,*

$$
\mathscr{A}(\mathbb{E}y) \cong \mathscr{A}(y). \tag{5.23}
$$

In particular, the grading is given by $\overline{\amalg}(\mathbb{E}y) \cong \overline{\amalg}(y)$.

PROOF. We will outline the main steps in the proof and indicate the additional results that can be obtained from this method. In order to accommodate the extension operator $\mathbb{E}$ acting on discrete braids we consider the parabolic recurrence relation $\mathbf{R}^\epsilon$ defined as $R^\epsilon_i = R_i$ for $i = 0, \cdots, d-1$ and $R^\epsilon_d = \epsilon^{-1}(x_{d+1} - x_d)$, and $R_0 = R_0(x_0, x_1)$. For $\epsilon > 0$ the parabolic flow is denoted by φ^ϵ on the augmented phase space $\widetilde{X}$, with $\widetilde{x} \in \widetilde{X}$ given by $\widetilde{x} = (x_0, \cdots, x_d) = (x, x_d)$. In the singular limit $\varphi^0 = \mathbb{E}\varphi$ is the induced parabolic flow on $\mathbb{E}X$. Consider the coordinates (x, z) with $z = x_{d+1} - x_d = x_0 - x_d$. With the reparametrization of time by $\tau = t/\epsilon$ we obtain the differential equations:

$$
\begin{aligned}
x' &= \epsilon X(x, z);\\
z' &= -z + \epsilon Z(x),
\end{aligned} \tag{5.24}
$$

where $X(x, z)$ and $Z(x)$ are given by: $X_i(x) = R_i(x_{i-1}, x_i, x_{i+1})$, with $i = 0, \cdots, d-2$, $X_{d-1}(x, z) = R_{d-1}(x_{d-2}, x_{d-1}, x_0 - z)$ and $Z(x) = R_0(x_0, x_1)$. The associated flow on $\widetilde{X}$ is denoted by φ^ϵ.

As for φ consider φ^ϵ with skeleton $\mathbb{E}y$ defined on the compact phase space $\widetilde{X}$. The construction in Section 5.3 of the Morse pre-order for φ^ϵ follows by the same token as before and we denote the Morse pre-order by $(\widetilde{X}, \leqslant^\dagger_\epsilon)$. The poset of partial equivalence classes is denoted by $(\mathsf{SC}_\epsilon, \leqslant_\epsilon)$. For $\epsilon > 0$ we obtain a tessellar phase diagram $(\amalg_\epsilon, \leqslant^\dagger_\epsilon)$ and a pure tessellar phase diagram $(\overline{\amalg}_\epsilon, \leqslant^\ddagger_\epsilon)$. Since the diagrams only depend on topological data they are independent of $\epsilon > 0$. The results in [24] imply that for every fiber $\pi^{-1}(y) \cap [x \text{ rel } y]_{\dot\sim}$ in $\mathscr{D}^d_{1,m}$ and $\pi^{-1}(\mathbb{E}y) \cap [\mathbb{E}x \text{ rel } \mathbb{E}y]_{\dot\sim}$ in $\mathscr{D}^{d+1}_{1,m}$ the Borel-Moore homologies are the same, cf. Thm. 5.32. The tessellar phase diagrams associated to the non-trivial homologies are denoted $\amalg(\mathbb{E}y)$ and $\overline{\amalg}(\mathbb{E}y)$. It also follows that for all classes in $[\widetilde{x}] \text{ rel } \mathbb{E}y \subset \pi^{-1}(\mathbb{E}y)$ for which $\pi^{-1}(\mathbb{E}y) \cap \mathbb{E}X = \varnothing$ the Borel-Moore homology is trivial, cf. [24, Rmk. 23]. These are exactly the classes that emerge when we extend via the operator $\mathbb{E}$. Indeed, if the maximal invariant set $\mathrm{Inv}([\widetilde{x}] \text{ rel } \mathbb{E}y) \neq \varnothing$ then the convergence properties of (5.24), cf. [24, Lem. 22], imply that there is a non-trivial invariant set for $\epsilon = 0$, which contradicts the fact that $\pi^{-1}(\mathbb{E}y) \cap \mathbb{E}X = \varnothing$. The pure tessellar phase diagram $\overline{\amalg}(\mathbb{E}y)$ obtained from $\amalg(\mathbb{E}y)$ remains unchanged since the non-trivial since a braid class $[x \text{ rel } y]_{\dot\sim}$ has non-trivial homology if and only if $[\mathbb{E}x \text{ rel } \mathbb{E}y]_{\dot\sim}$ has non-trivial homology. The partial order also remains unchanged. Indeed, by the above construction $(\amalg, \leqslant^\dagger) \hookrightarrow (\amalg_\epsilon, \leqslant^\dagger_\epsilon)$ and thus two elements in $\overline{\amalg}(y)$ are ordered if and only if they are ordered in $\overline{\amalg}(\mathbb{E}y)$, which proves that the pure tessellar phase diagram is invariant under the action of $\mathbb{E}$. By the same token the invariance of the homologies for convex sets in $\overline{\amalg}(y)$ are non-trivial if and only the homologies of the associated convex sets in $\overline{\amalg}(\mathbb{E}y)$ are non-trivial. By the nature of the induced connection matrix $\mathrm{d}^{\mathrm{para}}$ the homology braids are isomorphic and therefore the differential on C^{para} for $\mathbb{E}y$ can be chosen to be the same all $\epsilon \geqslant 0$. These facts combined prove the theorem. $\square$

The final result is to prove that the parabolic differential module is in fact an invariant of positive conjugacy classes of braid diagrams.

THEOREM 5.34. *The parabolic differential module $\mathscr{A}(\beta)$ of a (topological) positive braid β is a positive conjugacy class invariant.*

PROOF. Let $x \text{ rel } y \in \mathscr{D}^d_{1,m}$ and $x' \text{ rel } y' \in \mathscr{D}^{d'}_{1,m}$ be relative braids such that $\beta(x \text{ rel } y) \overset{+}{\sim} \beta(x' \text{ rel } y')$. By Proposition 5.28 we have that $\mathbb{E}^\ell x \text{ rel } \mathbb{E}^\ell y \sim \mathbb{E}^{\ell'} x \text{ rel } \mathbb{E}^{\ell'} y'$ for $\ell + d = \ell' + d'$ and ℓ, ℓ' sufficiently large. This implies

$$\begin{aligned}\vec{H}_{p,q}\big([x \text{ rel } y]_{\dot\sim}\big) &\cong \vec{H}_{p,q}\big([\mathbb{E}^\ell x \text{ rel } \mathbb{E}^\ell y]_{\dot\sim}\big) \cong \vec{H}_{p,q}\big([\mathbb{E}^{\ell'} x' \text{ rel } \mathbb{E}^{\ell'} y']_{\dot\sim}\big)\\ &\cong \vec{H}_{p,q}\big([x' \text{ rel } y']_{\dot\sim}\big),\end{aligned}$$

and therefore $\mathscr{A}(y) \cong \mathscr{A}(y')$. This justifies the notation $\mathscr{A}(\beta) := \mathscr{A}(y)$ with $\beta = \beta(y)$. $\square$

Since $\mathscr{A}(\beta)$ is an invariant for a positive braid β the associated reduced tessellar phase is denoted by $\big(\overline{\amalg}(\beta), \leqslant\big)$. From the tessellar phase diagram we can immediately derive the Poincare polynomial $\vec{P}_{\lambda,\mu}(\mathscr{A})$ by summing up the term $\vec{P}_{\lambda,\mu}$ in the tessellar phase diagram.

5.5.3. An example. For the skeletons y we use in this paper the strands y^- and y^+ do not intersect with the remaining strands in $\mathring{y}$, nor do they intersect with x. For that reason the convention is to label the basic words in $\mathscr{B}_m$ by σ_-

(intersections between y^- and y^1), σ_α, $\alpha = 1, \cdots, m-3$ (intersections between y^α and $y^{\alpha+1}$), and σ_+ (intersections between y^{m-2} and y^{m-1}). This implies that the words $\beta(y)$ in our setting consists only of the letters $\sigma_1, \cdots, \sigma_{m-3}$ and maybe regarded as a word $\beta(\mathring{y}) \in \mathscr{B}^+_{m-2}$.

In the examples below the words β are understood to be words $\beta(\mathring{y}) \in \mathscr{B}^+_{m-2}$. We compute the parabolic module $\mathscr{A}(\beta)$ and the pure tessellar phase diagrams for various positively conjugate representation of the braid $\beta = \sigma_1^2(\sigma_2\sigma_1)^2 \in \mathscr{B}_3^+$.

LEMMA 5.35. $\beta \stackrel{+}{\sim} \sigma_1^5\sigma_2$.

PROOF. From the braid group relations and positive conjugacy we have: $\beta = \sigma_1^2(\sigma_2\sigma_1)^2 = \sigma_1^2\sigma_2\sigma_1\sigma_2\sigma_1 \stackrel{+}{=} \sigma_1^3\sigma_2\sigma_1^2 \stackrel{+}{\sim} \sigma_1^5\sigma_2$. □

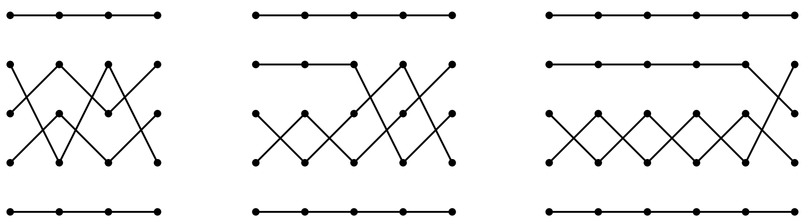

FIGURE 5.10. Three graphical representatives of positively conjugate braids whose words $\beta(\mathring{y})$ are given by $\sigma_2\sigma_1^2\sigma_2^2\sigma_1$, $\sigma_1^2(\sigma_2\sigma_1)^2$ and $\sigma_1^5\sigma_2$ respectively.

LEMMA 5.36. $\beta \stackrel{+}{\sim} \sigma_2\sigma_1^2\sigma_2^2\sigma_1$.

PROOF. As before: $\beta = \sigma_1^2\sigma_2\sigma_1\sigma_2\sigma_1 \stackrel{+}{=} \sigma_1^2\sigma_2^2\sigma_1\sigma_2 \stackrel{+}{\sim} \sigma_2\sigma_1^2\sigma_2^2\sigma_1$. □ □

Figure 5.10 depicts the three presentations of the braids conjugate to β (including β). The presentation $\sigma_2\sigma_1^2\sigma_2^2\sigma_1$ has the minimal number of discretization points. The braid $\sigma_2\sigma_1^2\sigma_2^2\sigma_1$ can be represented in $\mathscr{D}_5^3$, the braid $\sigma_1^2(\sigma_2\sigma_1)^2$ in $\mathscr{D}_5^4$ and $\sigma_1^5\sigma_2$ in $\mathscr{D}_5^5$. Figure 5.12 below shows the tessellar phase diagrams of $\sigma_2\sigma_1^2\sigma_2^2\sigma_1$ and $\sigma_1^2(\sigma_2\sigma_1)^2$ for $d = 3$ and $d = 4$ respectively. As expected the tessellated phase diagrams are not isomorphic since $y \in \mathscr{D}_5^4$ allows more relative braid classes. However, if we reduce the tessellated phase diagrams to only those with non-trivial Borel-Moore homology we obtain isomorphic posets $\overline{\Pi}\big(\sigma_2\sigma_1^2\sigma_2^2\sigma_1\big) \cong \overline{\Pi}\big(\sigma_1^2(\sigma_2\sigma_1)^2\big)$. Moreover the associated parabolic modules $\mathscr{A}$ for $\sigma_2\sigma_1^2\sigma_2^2\sigma_1$ and $\sigma_1^2(\sigma_2\sigma_1)^2$ are isomorphic, cf. Thm. 5.34.

$$0 \to \mathbb{Z}_2\langle \mathcal{S}_{15}\rangle \xrightarrow{\begin{pmatrix}1\end{pmatrix}} \mathbb{Z}_2\langle \mathcal{S}_{13}\rangle \xrightarrow{\begin{pmatrix}0\end{pmatrix}} \mathbb{Z}_2\langle \mathcal{S}_{8}\rangle \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \mathbb{Z}_2\langle \mathcal{S}_{4}\rangle \oplus \mathbb{Z}_2\langle \mathcal{S}_{0}\rangle \to 0$$

FIGURE 5.11. The parabolic differential module $\mathscr{A}(\beta)$, computed over $\mathbb{Z}_2$ coefficients, represented as chain complex.

If we apply the Morse relations in (4.17) we obtain

$$\vec{P}_{\lambda,\mu}(\mathscr{A}) = 2 + \lambda^2\mu + \lambda^3\mu^2 + \lambda^4\mu^3 = 1 + (1+\lambda\mu)\cdot\lambda^3\mu^2 + (1+\lambda^2\mu)\cdot 1,$$

which implies that $Q^1_{\lambda,\mu} = \lambda^3 s^2$ and $Q^2_{\lambda,\mu} = 1$. This provides information about the differentials $\mathrm{d}^1_{4,3}$ and $\mathrm{d}^2_{2,1}$. A straightforward but meticulous verification of the Morse relations in (4.17) show that the choices for Q^1 and Q^2 are unique. The ranks of $\mathrm{d}^1_{4,3}$ and $\mathrm{d}^2_{2,1}$ correspond to the ranks of the connection matrix $\mathrm{d}^{\mathrm{tile}}$.

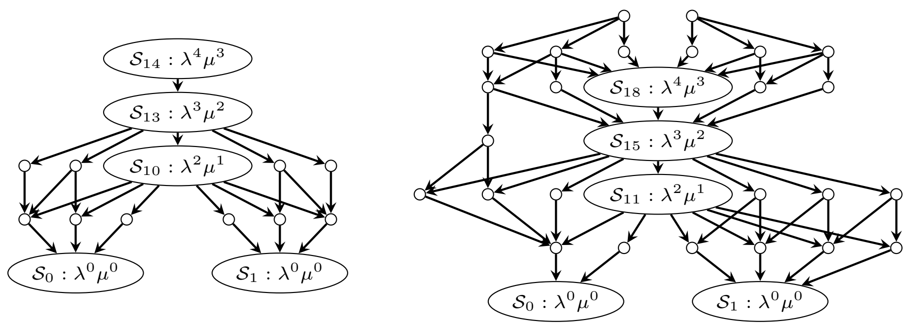

FIGURE 5.12. Tessellar phase diagrams for $\beta = \sigma_2\sigma_1^2\sigma_2^2\sigma_1$ [left] and $\beta = \sigma_1^2(\sigma_2\sigma_1)^2$ [right]. The posets SC for both examples are different but the posets $\overline{\amalg}(\sigma_2\sigma_1^2\sigma_2^2\sigma_1)$ and $\overline{\amalg}(\sigma_1^2(\sigma_2\sigma_1)^2)$ are isomorphic.

CHAPTER 6

Postlude

In this section we will address open questions and directions for further research based on the ideas in this text.

6.1. Topologization

In this section we comment on the general scheme of modeling dynamics as topology.

6.1.1. Variations on flow topologies. If we consider the *τ-forward-image* operator Γ^+_τ by considering forward images from $t \geqslant \tau > 0$ we obtain a derivative operator which is variation on Γ^+. The derivative Γ^+_τ is not idempotent in general. The fixed points of Γ^+_τ comprise the invariant sets of φ. The same can be done for $(-\tau)$-backward images which yields the operators $\Gamma^-_{-\tau}$. As for the block-flow topology we can also consider operators $\Gamma^{-\tau}_{\blacksquare} := \Gamma^-_{-\tau}\mathrm{cl}$ and the topologies $\mathscr{T}^{-\tau}_{\blacksquare}$, which are refinements of $\mathscr{T}^-_{\blacksquare}$. A different class of flow topologies can be derived by composing closure and forward image in reversed order, e.g. $\Gamma^{\blacksquare}_{-\tau} := \mathrm{cl}\,\Gamma^-_{-\tau}$ is a derivative operator on $\mathsf{Set}(X)$, and associated topologies $\mathscr{T}^{\blacksquare}_-$ and $\mathscr{T}^{\blacksquare}_{-\tau}$. If φ is continuous then $\Gamma^{\blacksquare}_\tau$ defines a derivative operator on $\mathsf{Set}(X)$. The τ-forward image derivative can be used to define the omega limit set operator $\Gamma_\omega U := \omega(U)$. A flow topology of particular interest is based on omega limit sets. Since $\omega(U)$ is forward invariant and closed it holds that $\omega(\omega(U)) \subset \omega(U)$. In addition, ω is a additive and $\omega(\varnothing) = \varnothing$, which proves that ω defines the derivative operator Γ_ω on $\mathsf{Set}(X)$. Therefore, $U \mapsto \mathrm{cl}_\omega U := U \cup \Gamma_\omega U$ is a closure operator and thus defines the topology $\mathscr{T}_\omega$ on X. Sets $U \subset X$ that are open in $\mathscr{T}$ and closed on $\mathscr{T}_\omega$ are exactly open attracting neighborhoods. It is sometimes convenient to consider trapping regions instead of attracting blocks. This can be achieved the operator: $U \mapsto \mathrm{cl}^+_\omega U := U \cup \Gamma^+ U \cup \Gamma_\omega U$.

6.1.2. Dynamical systems in the large. The flow topologies $\mathscr{T}^+$ and $\mathscr{T}^-$ and the derived flow topologies such as $\mathscr{T}^-_{\blacksquare}$ and $\mathscr{T}^{\blacksquare}_-$ inherit properties of the semi-flow φ at hand. For example if φ is trivial[1] then $\mathscr{T}^-_{\blacksquare} = \mathscr{T}^{\blacksquare}_- = \mathscr{T}$, and if X allows a dense flow φ then for instance $\overline{\mathscr{T}}_0$ is the trivial topology. In general, the flow topologies are not Hausdorff, independent of $\mathscr{T}$. A natural question to ask is which topologies are manifested as a derived flow topologies. In this setting we can think of a dynamical system in larger terms as a relation on X. This raises a

[1] i.e. $\varphi(t, x) = x$ for all $x \in X$ and for all $t \geqslant 0$.

deeper question whether we can equivalently study certain aspects of dynamical systems in the large as bi-topological spaces and bi-closure algebras. For the latter closure operators serve as a generalization of a dynamical system.

6.1.3. Beyond semi-flows. Most of the considerations in this paper carry over to discrete time dynamical systems such as iterating a map in X, cf. Remark 3.9. Another aspect that has played a minor role in the construction in this paper is the continuity of the dynamics with respect to the phase variable x. Finding discretizations, Morse pre-orders and Morse tessellation does not require continuity. The one instance where continuity plays a role is tying invariant dynamics to Morse tessellations. The latter uses algebraic topology and Wazewski's principle to conclude that non-trivial Borel-Moore homology yields non-trivial invariant dynamics. In particular, Wazewski's principle crucially uses the continuity of φ. For discrete time systems one can use a different notion of Conley index, cf. [58, 54, 57, 18].

6.2. Discretization

In terms of discretization, there are many intersetions with finite and combinatorial topology.

6.2.1. Quasi-isomorphic discretization and finite topologies. In practice (indeed, for all of the examples in this paper) the map $\mathrm{disc}\colon (X,\mathscr{T}) \to (\mathfrak{X},\leqslant)$ is a quasi-isomorphism, i.e. induces an isomorphism $H(X)\cong H(\mathfrak{X},\leqslant)$, where the latter is taken to be the singular homology of the finite topological space. In this case, as per Remark 4.26, one can discard disc and compute the graded tessellar complex $C^{\mathrm{part}}(\mathfrak{X})$ from $\mathrm{part}\colon \mathfrak{X}\to \mathsf{P}$ directly. This turns on the singular homology of $(\mathfrak{X},\leqslant)$ being easy to understand, which is often the case, e.g., disc is a CW-decomposition map. More generally, the singular homology of a finite topological space can be understood through a theorem of M. McCord [50], which says that there is a simplicial complex (the order complex) $K(\mathfrak{X})$ and a weak-homotopy equivalence $k\colon |K(\mathfrak{X})|\to\mathfrak{X}$. This can be used to induce a new map $\mathrm{part}_k\colon K(\mathfrak{X})\xrightarrow{k}\mathfrak{X}\to\mathsf{P}$ and form a graded chain complex $C^{\mathrm{part}_k}(K(\mathfrak{X}))$. As k is a weak homotopy equivalence (thus a quasi-isomorphism) $C^{\mathrm{part}_k}(K(\mathfrak{X}))$ has an isomorphic homology braid to $C^{\mathrm{part}}(\mathfrak{X})$. This provides another route for computation, and the role that finite topologies play seems worth examining.

6.2.2. Semi-conjugacies and finite models. A (bi-topological) discretization map $\mathrm{disc}\colon (X,\mathscr{T},\mathscr{T}_{\blacksquare}^{-}) \twoheadrightarrow (\mathfrak{X},\leqslant,\leqslant^0)$ is analogous to a semi-conjugacy in that one has a model system onto which the system of interest is mapped. There are situations in which it could be of interest to construct a semi-conjugacy of the semi-flow φ. The work of [49] provides results in this direction. If SC obeys a particular condition [2] and further conditions on the Conley indices and connection matrix $\mathrm{d}^{\mathrm{tile}}$

[2] Namely, what [7] terms a CW-poset, i.e. the face partial order of a regular CW complex.

are met,[3] then one can construct a surjective semi-conjugacy from X to $|K(\mathsf{SC})|$, the order complex associated to SC. Given that our starting point is tile, rather than the $C^{\mathrm{tile}}(X)$ itself, one has more data than assumed in [49] (in which only the nature of Conley indices and $\mathrm{d}^{\mathrm{tile}}$ is known), and it seems reasonable that these assumptions could be weakened.

6.3. Algebraization

There are further refinements and generalizations of the homological algebraic techniques we use to build algebraic models of dynamics.

6.3.1. Grading tessellar homology. In Remark 4.2.3 we explained via spectral sequences that one obtains a P-grading on tessellar homology in the case that P is a linear order. Applying this idea in the setting of parabolic recurrence relations yields the bi-graded parabolic homology. Via the bi-grading we obtain refined information about invariant sets for parabolic flows and connections between invariant sets. We have not fully explored the case of arbitrary partial orders P is this setting, i.e. grading tessellar homology by an arbitrary partial order. To do so we need to explore the theory of spectral systems, cf. [48]. The grading related to dynamics is denoted by p and the grading related to topology by q. In some case it is beneficial to disregard the q-grading and only consider the dynamical grading p.

6.3.2. Exact couple systems. In [48] Matschke introduces exact couple systems as a generalization of Massey's exact couples which allows generalizing spectral sequences to spectral systems. In Section 4.1 we discuss Franzosa's theory of connection matrices in terms of exact couple systems and Cartan-Eilenberg systems, cf. [63].

6.4. Braid and knot invariants

The application of discretization techniques to braids opens the door for investigating invariants in more general contexts.

6.4.1. Braid Floer homology. In Section 5.5 we discuss an invariant for positive conjugacy classes of braid in terms of a poset graded differential module. Our techniques in this paper use parabolic flows which restrict to positive braid diagrams and positive conjugacy. The general problem of braids is addressed in [66] and defines invariants for relative braid classes $x \text{ rel } y$ via Floer homology. One of the main results in [66] is that the homology invariants are isomorphic to the Floer homology of positive braids via Garside's normal form for braids. The latter makes that the Floer homology invariants for relative braids can be computed

[3] Termed H0-H3 in [49], these conditions ensure that $C^{\mathrm{tile}}(X)$ 'looks-like' the Morse complex of a gradient flow. Some of the braid examples from this text, after appropriate modification à la [40], satisfy these assumptions.

from the discrete invariants introduced in Section 5.5. The problem of formulating a differential module invariant in the general case for braids based on Floer homology is much harder. An extension of the results in [66] is to define a differential module for braids and show that it is an invariant of conjugacy classes of braids. A natural next step is to investigate the link to the discrete case in order to compute the differential module invariants. Another question in this setting is to understand how Markov moves impact the braid invariants which is important for investigating its relation with knot invariants.

6.4.2. Some immediate applications of the parabolic differential module. The theory of parabolic recurrence relations has been successfully applied to scalar parabolic differential equations of the type

$$u_t = u_{xx} + g(x, u, u_x).$$

cf. [25]. A collection of stationary solutions to the above equation is regarded as a continuous, positive braid β. As in the discrete case one may consider various relative braids $\alpha \text{ rel } \beta$. In particular, when α represents a single strand braid the analogy with the discrete theory is obtained by representing $\alpha \text{ rel } \beta$ as a piecewise linear braid $x \text{ rel } y$. The results in [25] make $\mathscr{A}(\beta)$ also an invariant for the above parabolic equation. The reduced tessellar phase diagram $\overline{\amalg}(\beta)$ yields stationary solutions for every vertex in $\overline{\amalg}(\beta)$. Moreover, in Conley index theory the boundary operator $\mathrm{d}^{\mathrm{para}}$ contains information about connecting orbits between stationary solutions, cf. [25, 65]. The highlight of the discrete theory is insight into the infinite dimensional Morse theory for the above parabolic equation.

The reduced tessellar phase diagram given by $\mathscr{A}(\beta)$ as provides detailed information about periodic points of surface diffeomorphism and diffeomorphisms of the 2-disc in particular. In [14] mapping classes of diffeomorphisms of the 2-disc are related to braids and positive braids in particular. To obtain a forcing theory for additional periodic points the Conley indices of braids $\alpha \text{ rel } \beta$ is computed. As in the previous example the reduced tessellar phase diagram given by $\mathscr{A}(\beta)$ forces new periodic points.

APPENDIX A

Binary relations and operators

In this appendix we summarize some elementary facts on binary relations and operators relevant for this text. We use [60] as are main reference for the theory of binary relations and [29] for operators.

A.1. Binary relations

Let X, Y be point sets. A *binary relation* is a subset $\phi \subset X \times Y$. If $X = Y$, then $\phi \subset X \times X$ is called an *endorelation*, or homogeneous (binary) relation on X. The top relation $X \times Y$ is denoted by $\top$ and the bottom relation $\varnothing$ by $\perp$. The identity is the diagonal $\mathrm{id} = \{(x, x) \mid x \in X\}$. The *opposite relation*, or *inverse relation* is denote by $\phi^{-1} = \{(y, x) \mid (x, y) \in \phi\} \subset Y \times X$. The *complement* ϕ^c is defined as $\phi^c := \{(x, y) \mid (x, y) \notin \phi\}$, which is the set-complement of ϕ. Some obvious properties are:

(i) $(\phi^{-1})^{-1} = \phi$;
(ii) $(\phi^{-1})^c = (\phi^c)^{-1}$;
(iii) $\phi \subset \psi$ if and only if $\phi^{-1} \subset \psi^{-1}$ if and only if $\phi^c \supset \psi^c$;
(iv) $(\phi \cup \psi)^{-1} = \phi^{-1} \cup \psi^{-1}$ and $(\phi \cap \psi)^{-1} = \phi^{-1} \cap \psi^{-1}$;
(v) $(\phi \cup \psi)^c = \phi^c \cap \psi^c$ and $(\phi \cap \psi)^c = \phi^c \cup \psi^c$ — De Morgan's laws.

An important operation on relations is *composition*: given $\phi \subset X \times Y$ and $\psi \subset Y \times Z$, then

$$\psi \cdot \phi := \big\{(x, z) \mid (x, y) \in \phi \text{ and } (y, z) \in \psi \text{ for some } y \in Y\big\} \subset X \times Z.$$

Some additional properties:

(vi) $(\psi \cdot \phi)^{-1} = \phi^{-1} \cdot \psi^{-1}$;
(vii) $\phi \subset \phi \cdot \phi^{-1} \cdot \phi$ and $\phi^{-1} \subset \phi^{-1} \cdot \phi \cdot \phi^{-1}$.

The set of all binary relations on $X \times Y$ is the power set $\mathsf{Set}(X \times Y)$, which is a complete and atomic Boolean algebra. In particular, for any $S \subset \mathsf{Set}(X \times Y)$:

$$\bigcup_{\phi \in S} \phi := \sup\big\{\phi \in S \mid S \subset \mathsf{Set}(X \times Y)\big\}, \tag{A.1}$$

is a well-defined relation in $\mathsf{Set}(X \times Y)$. The same applies the infima:

$$\bigcap_{\phi \in S} \phi := \inf\big\{\phi \in S \mid S \subset \mathsf{Set}(X \times Y)\big\}, \tag{A.2}$$

is a well-defined relation in $\mathsf{Set}(X \times Y)$.

A binary relation ϕ is *left total* if for all $x \in X$ there exists an $y \in Y$ such that $(x, y) \in \phi$. A binary relation ϕ is *right total*, or *surjective* if for all $y \in Y$ there exists an $x \in X$ such that $(x, y) \in \phi$, i.e. ϕ^{-1} is left total. Composition with the opposite relation yields the following properties:

(viii) ϕ is left total if and only if $\mathrm{id} \subset \phi^{-1} \cdot \phi$;
(ix) ϕ is right total if and only if $\mathrm{id} \subset \phi \cdot \phi^{-1}$.

Binary relations come with many different properties. In this text partial orders and pre-orders are special cases of homogeneous binary relations on X that play a pivotal role in the theory. For convenience we now consider homogeneous relations on X. Instead of using the latter terminology we will refer to homogeneous binary relations as binary relations on X, which are elements of the Boolean algebra Set$(X \times X)$. A selection of special properties of binary relations on X are:

- *reflexive* if $\mathrm{id} \subset \phi$;
- *irreflexive* if $\mathrm{id} \subset \phi^c$;
- *symmetric* if $\phi = \phi^{-1}$;
- *asymmetric* if $\phi^{-1} \subset \phi^c$, i.e. $(x, y) \in \phi$ implies $(y, x) \notin \phi$;
- *anti-symmetric* if $\phi \cap \phi^{-1} \subset \mathrm{id}$, i.e. $(x, y) \in \phi$ and $(y, x) \in \phi$ implies $x = y$;
- *transitive* if $\phi^2 \subset \phi$, i.e. $(x, y), (y, z) \in \phi$ implies that $(x, z) \in \phi$;
- *dense* if $\phi \subset \phi^2$, i.e. for every $(x, y) \in \phi$ there exists a $z \in X$ such that $(x, z), (z, y) \in \phi$.

In particular, every reflexive relation is dense and both left and right total. With the above properties one can indicate a number of common binary relations on X. A binary relation ϕ on X is a(n):

- *pre-order* if ϕ is reflexive and transitive, i.e. $\mathrm{id} \subset \phi$ and $\phi^2 = \phi$;
- *partial order* if ϕ is reflexive, anti-symmetric and transitive, i.e. $\mathrm{id} \subset \phi$, $\phi \cap \phi^{-1} \subset \mathrm{id}$ and $\phi^2 = \phi$;
- *strict partial order* if ϕ is irreflexive and transitive, which is equivalent to asymmetric and transitive;
- *linear order*, or *total order* if ϕ is a partial such that $(x, y) \in \phi$, or $(y, x) \in \phi$ for all pairs $(x, y) \in X \times Y$ (the last conditions equivalent to $\phi \cup \phi^{-1} = \top$);
- *equivalence relation* if ϕ is reflexive, transitive and symmetric.

For a binary relation on X we use different notations: $(x, y) \in \phi$, which is equivalent to $x \,\phi\, y$. In particular for partial orders and pre-orders we write $x \leqslant y$, or $x \leqslant_\phi y$. For equivalence relations we use $x \sim y$, or $x \sim_\phi y$. Given a partial order we can write the associated strict order and vice verse, i.e. given a partial order ϕ, then $\phi_\bullet := \phi \cap \mathrm{id}^c$ is the associated strict partial order, and given a strict partial order ψ, then $\psi^\bullet := \psi \cup \mathrm{id}$ is the associated partial order. This yields the correspondences:

$$(\phi_\bullet)^\bullet = \phi, \qquad (\psi^\bullet)_\bullet = \psi.$$

A *Hasse relation*, or *Hasse diagram* for a partial order ϕ is defined as:

$$\phi_H := \phi_\bullet \cap (\phi_\bullet^2)^c,$$

and corresponds to the usual notion of Hasse diagram for partial orders on a finite set X. Transitive reflexive closure, which we now explain, of the Hasse relation retrieves the partial order.

Given a binary relation ϕ on X, then its *transitive closure* is defined as

$$\phi^{+} := \inf\{\psi \subset X \times X \mid \phi \subset \psi,\ \psi^2 \subset \psi\} = \bigcup_{k \geqslant 1} \phi^k.$$

The notion of *transitive reduction* only makes sense for finite sets X but is not well-defined for infinite sets in general. The *transitive reflexive closure* is defined as

$$\phi^{+=} := \inf\{\psi \subset X \times X \mid \mathrm{id} \cup \phi \subset \psi,\ \psi^2 \subset \psi\} = \bigcup_{k \geqslant 0} \phi^k.$$

The transitive reflexive closure $\phi^{+=}$ of a binary relation ϕ is pre-order on X. Via *reflexive closure* $\phi^{=} := \mathrm{id} \cup \phi$ we have that $\phi^{+=} = \mathrm{id} \cup \phi^{+}$. *Reflexive reduction* is defined for all binary relations and is given as $\phi^{\neq} := \phi \cap \mathrm{id}^c$. Finally, the *strongly connected components* of a binary relation on X are given by the equivalence relation

$$\phi_{\mathsf{SC}} := \phi^{+=} \cap \left(\phi^{+=}\right)^{-1}, \tag{A.3}$$

whose equivalence classes are the strongly connected components of ϕ. The latter is of particular importance for binary relations on finite sets (digraphs), cf. Rem. 2.24. The pre-order $\phi^{+=}$ yields a partial order on the strongly connected components, cf. App. B.1 (ordered tessellations).

A.2. Modal operators

Related to binary relations $\phi \subset X \times Y$ is the notion of operator on the algebra of subsets of X and Y. Let $\phi \subset X \times Y$ be a binary relation. Then, for $U \subset X$ define

$$\phi\, U := \{y \in Y \mid (x, y) \in \phi \ \text{ for some } x \in U\} = \bigcup_{x \in U} \phi\, x \subset Y,$$

where $\phi\, x = \{y \in Y \mid (x, y) \in \phi\}$. By definition $\phi\, \varnothing = \varnothing$ and $\phi(U \cup U') = \phi\, U \cup \phi\, U'$, which show that ϕ regarded as operator is a *modal operator* from $\mathsf{Set}(X)$ to $\mathsf{Set}(Y)$. In this text the operators are mostly from $\mathsf{Set}(X)$ to $\mathsf{Set}(X)$. The following properties follow from the definition of operator:

(i) $\phi\big(\bigcup_{i\in I} U_i\big) = \bigcup_{i \in I} \phi\, U_i$;
(ii) $\phi\big(\bigcap_{i\in I} U_i\big) \subset \bigcap_{i \in I} \phi\, U_i$, both for arbitrary families $\{U_i\}_{i\in I}$ of subsets in X.

The modal operator defined by a binary relation are completely additive. The opposite relation ϕ^{-1} regarded as operator is related to ϕ as operator in a similar way as the inverse function to a function. First of all (i)-(ii) also holds for ϕ^{-1} as operator. Moreover,

(iii) $U \cap \phi^{-1} Y \subset \phi^{-1}\big(\phi\, U\big)$, for all $U \subset X$;
(iv) $V \cap \phi\, X \subset \phi\big(\phi^{-1} V\big)$, for all $V \subset Y$.

REMARK A.1. If ϕ is left total then A.2(iii) corresponds to A.1(viii), and A.2(iv) corresponds to A.1(ix). Note that if $V = \phi U$ in A.2(iv), then the identity A.1(vii) is satisfied. The same holds for A.2(iii).

To prove (iii) and (iv) we argue as follows. The set $\phi^{-1}Y = \{x \in X \mid (y,x) \in \phi^{-1} \text{ for some } y \in Y\} = \{x \in X \mid (x,y) \in \phi \text{ for some } y \in Y\}$ is the domain X for which $\phi x \neq \varnothing$. Futhermore,

$$\begin{aligned}\phi U &= \{y \in Y \mid (x,y) \in \phi \text{ for some } x \in U\}\\ &= \{y \in Y \mid (x,y) \in \phi \text{ for some } x \in U \cap \phi^{-1}Y\} = \phi(U \cap \phi^{-1}Y),\end{aligned}$$

and thus ϕ may be regarded as a left total relation on $\phi^{-1}Y \times Y$. For any subset $U \subset X$, $U \cap \phi^{-1}Y$ is a subset of $\phi^{-1}Y$. By A.1(viii) this gives

$$U \cap \phi^{-1}Y \subset \phi^{-1}\big(\phi(U \cap \phi^{-1}Y)\big) = \phi^{-1}(\phi U),$$

which proves (iii). Statement (iv) is proved by changing the role of ϕ and ϕ^{-1}.

A useful identity for $\phi^{-1}U$ is given by

$$\phi^{-1}V = \{x \in X \mid \phi x \cap V \neq \varnothing\}, \quad V \subset Y. \tag{A.4}$$

Indeed,

$$\begin{aligned}\phi^{-1}V &= \{x \in X \mid (y,x) \in \phi^{-1} \text{ for some } y \in V\}\\ &= \{x \in X \mid (x,y) \in \phi \text{ for some } y \in V\}\\ &= \{x \in X \mid y \in \phi x \text{ for some } y \in V\}\\ &= \{x \in X \mid \phi x \cap V \neq \varnothing\}.\end{aligned}$$

As operator $\bar{\phi} := \phi^{-1}$ is also referred to as *conjugate operator*. Of other crucial importance is the *dual operator* related to ϕ. Define,

$$\phi^* U := \big(\phi U^c\big)^c, \quad U \subset X. \tag{A.5}$$

REMARK A.2. In general this definition works for modal operators on Boolean algebras, not just completely additive operators as discussed in this appendix. For example int is the dual operator of cl on $\mathsf{Set}(X)$. If cl is defined via ϕ, i.e. Alexandrov topology, then the conjugate operator is star, cf. Sect. 2.3.1.

By the same token we define the dual of ϕ^{-1}:

$$\phi^{-*} V := \big(\phi^{-1} V^c\big)^c, \quad V \subset Y. \tag{A.6}$$

Useful identities in this setting are:

(v) $\phi^* U^c = \big(\phi U\big)^c$ and $\big(\phi^* U\big)^c = \phi U^c$ for all $U \subset X$;
(vi) $\phi^{-*} V^c = \big(\phi^{-1} V\big)^c$ and $\big(\phi^{-*} V\big)^c = \phi^{-1} V^c$ for all $V \subset Y$.

As for ϕ^{-1} there is a convenient characterization of ϕ^{-*}:

$$\phi^{-*} V = \{x \in X \mid \phi x \subset V\}, \quad V \subset Y. \tag{A.7}$$

Indeed, by (A.4) and (A.6)

$$\phi^{-*}V = \big(\phi^{-1}V^c\big)^c = \{x \in X \mid \phi x \cap V^c \neq \varnothing\}^c = \{x \in X \mid \phi x \subset V\},$$

which proves (A.7). Note that by the latter characterization we have $\phi^{-*}V \subset \phi^{-1}V$ for all $V \subset Y$. This gives:

(vii) $\big(\phi^{-*}V\big)^c \supset \phi^{-*}V^c$;

(viii) $\left(\phi^{-1}V\right)^c \subset \phi^{-1}V^c$, both for all $V \subset Y$.

REMARK A.3. The characterizations in (A.4) and (A.7) also hold for ϕ and ϕ^* by simply replacing ϕ by ϕ^{-1}. This yields the same identities in (vii)-(viii) for ϕ and ϕ^*.

The modal operators defined by binary relations are completely additive. As a consequence of the definition of dual the latter are *completely multiplicative*.

(ix) $\phi^*\left(\bigcup_{i\in I} U_i\right) \supset \bigcup_{i\in I} \phi^* U_i$;
(x) $\phi^*\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} \phi^* U_i$, both for arbitrary families $\{U_i\}_{i\in I}$ of subsets in X.

The same holds for ϕ^{-*}. For example (x) is derived as follows:

$$\phi^*\Big(\bigcap_{i\in I} U_i\Big) = \Big(\phi\big(\bigcap_{i\in I} U_i\big)^c\Big)^c = \Big(\phi\big(\bigcup_{i\in I} U_i^c\big)\Big)^c = \Big(\bigcup_{i\in I} \phi\, U_i^c\Big)^c = \bigcap_{i\in I} \big(\phi\, U_i^c\big)^c = \bigcap_{i\in I} \phi^* U_i.$$

Due to the definition of dual there are additional properties with respect to composition in contrast to (iii) and (iv):

(xi) $U \subset \phi^{-*}\big(\phi U\big)$, for all $U \subset X$;
(xii) $V \supset \phi\big(\phi^{-*}V\big)$, for all $V \subset Y$.

To prove (xi) observe that

$$\phi^{-*}\big(\phi U\big) = \big\{x \in X \mid \phi\, x \subset \phi U\big\} \supset U.$$

As for (xii) we have:

$$\begin{aligned} \phi\big(\phi^{-*}V\big) &= \big\{y \in Y \mid (x,y) \in \phi \ \text{ for some } x \in \phi^{-*}V\big\} \\ &= \big\{y \in Y \mid (x,y) \in \phi \ \text{ for some } x \text{ such that } \phi\, x \subset V\big\} \subset V, \end{aligned}$$

i.e. $y \in \phi\, x \subset V$, which proves (xii).

REMARK A.4. In the special case that ϕ is given by a function $f\colon X \to Y$, i.e. $\phi = \{(x,y) \mid y = f(x)\}$, then the opposite relation ϕ^{-1} is given by: $\phi^{-1} = \{(y,x) \mid y = f(x)\}$. As operators we have $\phi^{-1}V$ and $\phi^{-*}V$ and, since $\{f(x)\}$ is a singleton set,

$$\phi^{-1}V = \big\{x \in X \mid \{f(x)\} \cap V \neq \varnothing\big\} = \big\{x \in X \mid \{f(x)\} \subset V\big\} = \phi^{-*}V.$$

We write $= f^{-1}V = \phi^{-1}V = \phi^{-*}V$. For composition this implies by (iv) and (xii):

$$f(X) \cap V = \phi\, X \cap V \subset \phi(\phi^{-1}V) = \phi(\phi^{-*}V) \subset V,$$

and therefore $f(X) \cap V \subset f(f^{-1}V) \subset V$, which yields identity exactly when f is surjective. Similarly we have $U \subset f^{-1}f(U)$.

REMARK A.5. Properties of binary relations imply properties on operators. For instance an operator that satisfies $\phi^2 U \subset \phi U$ for all $U \subset X$, corresponds to a transitive relation.

A.3. Duality

In this section we recall the duality for binary relations on X and operators on $\mathsf{Set}(X)$. The main source of reference for this section is [27]. The prime example of a complete and atomic Boolean algebra is the power set of a point set X denoted by $\mathsf{Set}(X)$. The latter is a Boolean algebra with respect to intersection, union and complement, and is closed under arbitrary intersections and unions, and every element is the union of atoms $\{x\}$, $x \in X$. Let Y be another point set. A Boolean homomorphism is a lattice homomorphism that preserves the units, i.e. $\varnothing$ and X. A Boolean homomorphism $\Phi\colon \mathsf{Set}(Y) \to \mathsf{Set}(X)$ between the complete and atomic Boolean algebras $\mathsf{Set}(Y)$ and $\mathsf{Set}(X)$ is *completely additive* if

$$\Phi\Big(\bigcup_{i\in I} U_i\Big) = \bigcup_{i\in I} \Phi U_i, \tag{A.8}$$

for arbitrary unions $\bigcup_{i\in I} U_i$, $U_i \subset X$. Since Φ is Boolean the above complete additivity is equivalent to

$$\Phi\Big(\bigcap_{i\in I} U_i\Big) = \bigcap_{i\in I} \Phi U_i, \tag{A.9}$$

for arbitrary intersections $\bigcap_{i\in I} U_i, U_i \subset X$. Related to Φ we define a binary relation on $X \times Y$:

$$(x, y) \in \phi \quad \text{if and only if} \quad x \in \Phi\{y\}. \tag{A.10}$$

LEMMA A.6. *$\Phi = \phi^{-1}$ as operators from $\mathsf{Set}(Y)$ to $\mathsf{Set}(X)$.*

PROOF. By defintion

$$(x, y) \in \phi \quad \text{if and only if} \quad (y, x) \in \phi^{-1} \quad \text{if and only if} \quad x \in \phi^{-1}\{y\}.$$

This implies that $\Phi\{y\} = \phi^{-1}\{y\}$ for all $y \in Y$. By the complete additivity of Φ we have:

$$\phi^{-1}V = \phi^{-1}\Big(\bigcup_{y\in V}\{y\}\Big) = \bigcup_{y\in V}\phi^{-1}\{y\} = \bigcup_{y\in V}\Phi\{y\} = \Phi V, \quad V \subset Y.$$

Moreover, $\phi^{-1}\varnothing = \Phi\varnothing = \varnothing$, which completes the proof. □

In the Boolean setting the properties on Φ yield a restriction on ϕ as is displayed in the following result.

PROPOSITION A.7. *If $\Phi\colon \mathsf{Set}(Y) \to \mathsf{Set}(X)$ is completely additive Boolean homomorphism, then $\Phi = \phi^{-1}$, where*

$$\phi = \big\{(x, f(x) \mid f\colon X \to Y\big\},$$

and $(x, y) \in \phi$ uniquely determines $f(x) = y$. We say that the binary relation ϕ is functional and we write $\Phi = f^{-1}$.

PROOF. For $y \neq y'$ we have $\Phi\{y\} \cap \Phi\{y'\} = \Phi(\{y\} \cap \{y'\}) = \Phi\varnothing = \varnothing$. Also $X = \Phi Y = \Phi(\bigcup\{y\}) = \bigcup \Phi\{y\}$, which uses the complete additivity. This implies that given $x \in X$, then there exists a unique $y \in Y$ such that $x \in \Phi\{y\}$ since the sets $\Phi\{y\}$ are disjoint for distinct $y \in Y$. The latter defines the functional relation ϕ and $f(x) = y$, i.e. the points in ϕ are given by the pairs $(x, f(x))$. $\square$

REMARK A.8. The definition of ϕ related to Φ in (A.10) is a choice that will be crucial in our treatment of duality. Some authors use (A.10) to define ϕ^{-1}. Our choice in consistent with the conventions in the logics liturature, cf. [27]. A more compelling reason for the above choice is Proposition A.7:

$$\Phi\colon \mathsf{Set}(Y) \to \mathsf{Set}(X) \quad \Longleftrightarrow \quad \Phi = \phi^{-1}, \text{ with } \phi = \big\{(x, f(x) \mid f\colon X \to Y\big\},$$

which characterizes all completely additive Boolean homomorphisms.

The notion of Boolean homomorphism can be weakened to the notion of modal operator. Let $\Phi\colon \mathsf{Set}(Y) \to \mathsf{Set}(X)$ be a *completely additive modal operator*, i.e. Φ satisfies $\Phi\varnothing = \varnothing$ and the additivity condition in (A.8). By the definition in (A.10) and Lemma A.6 every completely additive operator Φ uniquely determines a binary relation $\phi \subset X \times Y$. However, since Φ is not necessarily Boolean the relation ϕ is not necessarily functional. Instead, we obtain any binary relation on $X \times Y$. We use the notation:

$$\Phi = \phi^{-1}\colon \mathsf{Set}(Y) \to \mathsf{Set}(X), \quad \text{and} \quad \phi = \Phi^{-1} \subset X \times Y. \tag{A.11}$$

In particular,

$$(\phi^{-1})^{-1} = \phi,, \quad \text{and} \quad (\Phi^{-1})^{-1} = \Phi. \tag{A.12}$$

The duality between operators and relations explained in this appendix applies to complete and atomic Boolean algebras. In [38] and [30] this duality is extended to arbitrary Boolean algebras and a special classes of binary relations — *Boolean relations*. In the case of a single Boolean algebra with (modal) operator $(\mathsf{B}, \mathbf{c})$ (not necessarily completely additive) is embedded in a complete and atomic Boolean algebra with with a completely additive extension of $\mathbf{c}$.

APPENDIX B

Order Theory

In this appendix we outline some of the most prominent concepts of order theory that are used in this text. Our main references are [15], [59] and [38].

B.1. Posets and pre-orders

In Appendix A we defined binary relations and in particular partial orders and pre-orders. In the setting of finite sets we will focus here on finite partially ordered sets, or *posets* which we will denote by $(\mathsf{P}, \leqslant)$, where P is a finite set and $\leqslant$ a partial order (or pre-order). If there is no ambiguity on the partial order we typically denote a finite poset by P, or Q. A function $\nu\colon \mathsf{P} \to \mathsf{Q}$ between finite posets is *order-preserving* if $p \leqslant q$ implies that $\nu(p) \leqslant \nu(q)$.

The *category of finite posets*, denoted **FPoset**, is the category whose objects are finite posets and whose morphisms are order-preserving maps. The category of finite pre-orders is denoted by **FPreOrd**.

Let P be a finite poset. An *up-set* of P is a subset $I \subset \mathsf{P}$ such that if $p \in \mathsf{U}$ and $p \leqslant q$ then $q \in I$. For $p \in \mathsf{P}$ the *up-set at* p is $\uparrow p := \{q \in \mathsf{P} : p \leqslant q\}$ which is also called a *principal up-set*. Following [42], we denote the collection of up-sets by $\mathsf{U}(\mathsf{P})$. A *down-set* of P is a set $I \subset \mathsf{P}$ such that if $q \in D$ and $p \leqslant q$ then $p \in I$. The *down-set at* q is $\downarrow q := \{p \in \mathsf{P} : p \leqslant q\}$ which is called a *principal down-set*. Following [42], we denote the collection of down-sets by $\mathsf{O}(\mathsf{P})$.

For $p, q \in \mathsf{P}$ the *interval* from p to q, denoted $[p, q]$, is the set $\{r \in \mathsf{P} : p \leqslant r \leqslant q\}$. A subset $I \subset \mathsf{P}$ is *convex* if whenever $p, q \in I$ then $[p, q] \subset I$. Every convex set is of the form $\alpha \smallsetminus \beta$ with $\alpha, \beta \in \mathsf{O}(\mathsf{P})$. We denote the collection of convex sets by $\mathsf{Co}(\mathsf{P})$. Every convex set of P can be obtained as intersection of a down-set and an up-set. Under a poset morphism the preimage of a convex set is a convex set, cf. [59].

If P is a pre-order, then the equivalence classes are ordered as $[p] \leqslant [q]$ if and only if $p \leqslant q$. The poset of equivalence class $\mathsf{P}/_\sim$ is called an *ordered tessellation* for P. The latter is also referred to as an *ordered partition* of P

B.2. Lattices

Some texts introduce lattices as a particular type of poset. Instead, we begin with the definition of lattice as an algebraic structure. For a discussion of the relationship of these two definitions the reader may consult [15, Chapter 2], in particular [15, Theorem 2.9].

DEFINITION B.1. A *lattice* is a set L with the binary operations $\vee, \wedge \colon \mathsf{L} \times \mathsf{L} \to \mathsf{L}$ satisfying the following four axioms:

(i) $a \wedge a = a \vee a = a$ for all $a \in \mathsf{L}$ (idempotence);
(ii) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ for all $a, b \in \mathsf{L}$ (commutativity);
(iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$ for all $a, b, c \in \mathsf{L}$ (associativity);
(iv) $a \wedge (a \vee b) = a \vee (a \wedge b) = a$ for all $a, b \in \mathsf{L}$ (absorption law).

A lattice L has an associated poset structure given by $a \leqslant b$ if $a = a \wedge b$ or, equivalently, if $b = a \vee b$. A lattice L is *distributive* if it satisfies the additional axiom:

(v) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in \mathsf{L}$ (distributivity);

A lattice L is *bounded* if there exist *neutral* elements 0 and 1 that satisfy the following property:

(vi) $0 \wedge a = 0, 0 \vee a = a, 1 \wedge a = a, 1 \vee a = 1$ for all $a \in \mathsf{L}$.

A *complemented lattice*, also called a *Boolean algebra*, is a bounded lattice (with least element 0 and greatest element 1), in which every element a has a complement, i.e. an element b such that $a \vee b = 1$ and $a \wedge b = 0$.

A lattice homomorphism $f \colon \mathsf{L} \to \mathsf{M}$ is a map such that if $a, b \in \mathsf{L}$ then $f(a \wedge b) = f(a) \wedge f(b)$ and $f(a \vee b) = f(a) \vee f(b)$. If L and M are bounded lattices then we also require that $f(0) = 0$ and $f(1) = 1$. In particular, we are interested in finite lattices. Every finite lattice is bounded. A subset $\mathsf{K} \subset \mathsf{L}$ is a sublattice of L if $a, b \in \mathsf{K}$ implies that $a \vee b \in \mathsf{K}$ and $a \wedge b \in \mathsf{K}$. For sublattices of bounded lattices we impose the extra condition that $0, 1 \in \mathsf{K}$. The *category of finite distributive lattices*, denoted **FDLat**, is the category whose objects are finite distributive lattices and whose morphisms are lattice homomorphisms.

An element $a \in \mathsf{L}$ is *join-irreducible* if it has a unique *immediate predecessor*; given a join-irreducible a we denote its unique predecessor by $a^{\blacktriangleleft}$. The set of join-irreducible elements of L is denoted by $\mathsf{J}(\mathsf{L})$. The join-irreducible elements form a poset $(\mathsf{J}(\mathsf{L}), \leqslant)$, where the order $\leqslant$ is the restriction of the partial order of L.

B.3. Birkhoff duality

Given a finite distributive lattice L, $\mathsf{J}(\mathsf{L})$ is a poset with respect to set-inclusion. Conversely, given a finite poset $(\mathsf{P}, \leqslant)$ the set of downsets $\mathsf{O}(\mathsf{P})$ is a bounded distributive lattice under $\wedge = \cap$ and $\vee = \cup$. The following theorem often goes under the moniker 'Birkhoff's Representation Theorem' and the duality will be referred to as *Birkhoff duality*.

THEOREM B.2 (cf. [59], Theorem 10.4 and [40]). *The applications* $\mathsf{L} \Rightarrow \mathsf{J}(\mathsf{L})$ *and* $\mathsf{P} \Rightarrow \mathsf{O}(\mathsf{P})$ *are contravariant functors from* **FDLat** *to* **FPoset** *and from* **FPoset** *to* **FDLat** *respectively. A lattice homomorphism* $h \colon \mathsf{K} \to \mathsf{L}$ *is dual to an order-preserving map* $\mathsf{J}(h) \colon \mathsf{J}(\mathsf{L}) \to \mathsf{J}(\mathsf{K})$ *and an order-preserving map* $\nu \colon \mathsf{P} \to \mathsf{Q}$ *is dual to a lattice homomorphism* $\mathsf{O}(\nu) \colon \mathsf{O}(\mathsf{Q}) \to \mathsf{O}(\mathsf{P})$ *given by the formulas This may be represent this via the* *following diagram:*

$$
\begin{tikzcd}
\mathsf{K} \arrow[d, "h"'] & & \mathsf{J}(\mathsf{K}) & \mathsf{P} \arrow[d, "\nu"'] & & \mathsf{O}(\mathsf{P}) \\
\mathsf{L} \arrow[r, phantom, "\overset{\mathsf{J}}{\Longrightarrow}"] & & \mathsf{J}(\mathsf{L}) \arrow[u, "\mathsf{J}(h)"'] & \mathsf{Q} \arrow[r, phantom, "\overset{\mathsf{O}}{\Longrightarrow}"] & & \mathsf{O}(\mathsf{Q}) \arrow[u, "\mathsf{O}(\nu)"']
\end{tikzcd}
$$

$$
\begin{aligned}
\mathsf{J}(h)(a) &= \min h^{-1}\left(\uparrow a\right), && \textit{where } a \in \mathsf{J}(\mathsf{L}),\\
\mathsf{O}(\nu)(\alpha) &= \nu^{-1}(\alpha), && \textit{where } \alpha \in \mathsf{O}(\mathsf{Q}),
\end{aligned}
$$

respectively. Furthermore,

$$\mathsf{L} \cong \mathsf{O}(\mathsf{J}(\mathsf{L})) \quad \textit{and} \quad \mathsf{P} \cong \mathsf{J}(\mathsf{O}(\mathsf{P})).$$

The pair of contravariant functors O and J called the *Birkhoff transforms*. Given $\nu\colon \mathsf{P} \to \mathsf{Q}$ we say that $\mathsf{O}(\nu)$ is the *Birkhoff dual* to ν. Similarly, for $h\colon \mathsf{K} \to \mathsf{L}$ we say that $\mathsf{J}(h)$ is the *Birkhoff dual* to h. A lattice homomorphism h is injective if and only if $\mathsf{J}(h)$ is surjective, and h is surjective if and only if $\mathsf{J}(h)$ is an order-embedding.

REMARK B.3. If $\mathsf{L} = \mathsf{O}(\mathsf{P})$ and $\mathsf{K} = \mathsf{O}(\mathsf{Q})$ then the homomorphism $\mathsf{J}(h)\colon \mathsf{P} \to \mathsf{Q}$ is given by the formula $\mathsf{J}(h)(p) = \min\left\{q \in \mathsf{Q} \mid p \in h\left(\downarrow q\right)\right\}$, cf. [15, Thm. 5.19].

APPENDIX C

Grading, filtering and differential modules

In this appendix we explain gradings and filterings in the context order theory applied to sets and modules In this setting we discuss the duality between gradings and filterings in the spirits of Birkhoff's representation theorem.

C.1. P-gradings and $\mathsf{O}(\mathsf{P})$-filterings

We start with the definitions of grading and filtering on a set X which carry over to the setting of modules in Appendix C.2.

DEFINITION C.1. An *ordered tessellation*, or *ordered partition* of a set X is a poset $(\mathsf{T}, \leqslant)$ consisting of non-empty subsets $T \subset X$, such that

(i) $T \cap T' = \varnothing$ for all distinct $T, T' \in \mathsf{T}$;
(ii) $\bigcup_{T\in\mathsf{T}} T = X$.

One can also consider more general binary relations on tessellations such as preorders.

Given the set of down-sets $\mathsf{O}(\mathsf{T})$ in $(\mathsf{T}, \leqslant)$ we define the lattice:

$$\mathsf{N}(\mathsf{T}) := \Big\{ N \subset X \mid N = \bigcup_{T\in\alpha} T,\ \alpha \in \mathsf{O}(\mathsf{T}) \Big\},$$

which is a finite sublattice of $\mathsf{Set}(X)$ with binary operations $\cap$ and $\cup$.

DEFINITION C.2. A finite sublattice $\mathsf{N} \subset \mathsf{Set}(X)$ is called a *filtering*[1] on X.

The sublattice $\mathsf{N}(\mathsf{T})$ is a filtering on X, constructed from T. If we start with a filtering $\mathsf{N} \subset \mathsf{Set}(X)$ we construct an ordered tessellation from N as follows. For $N \in \mathsf{J}(\mathsf{N})$ define $T := N \smallsetminus N^{\blacktriangleleft}$, where $N^{\blacktriangleleft}$ is the unique immediate predecessor of N in the lattice N. From [44] it follows that for distinct $N, N' \in \mathsf{J}(\mathsf{N})$, then (i) $T, T' \neq \varnothing$, (ii) $T \cap T' = \varnothing$, and (iii) $\bigcup_{N\in\mathsf{J}(\mathsf{N})} N \smallsetminus N^{\blacktriangleleft} = X$. We order the tiles T as: $T \leqslant T'$ if and only if $N \subseteq N'$. This makes $\big(\mathsf{T}(\mathsf{N}), \leqslant\big) \cong (\mathsf{J}(\mathsf{N}), \subset)$ an ordered tessellation of X. By Birkhoff duality we conclude:

$$\mathsf{N}\big(\mathsf{T}(\mathsf{N})\big) = \mathsf{N} \quad \text{and} \quad \mathsf{T}\big(\mathsf{N}(\mathsf{T})\big) = \mathsf{T}.$$

Filterings regarded as finite sublattices of subsets in X provide an algebraic point of view for the construction ordered tessellations of X. An $\mathsf{O}(\mathsf{P})$-*filtering* on X is a

[1] We use term *filtering* for an arbitrary sublattice of subsets in $\mathsf{Set}(X)$ as opposed to *filtration* which is commonly used for a linearly ordered sublattice.

lattice homomorphism

$$\mathrm{flt}\colon \mathsf{O}(\mathsf{P}) \to \mathsf{Set}(X),$$

and the pair (X, flt) is called an $\mathsf{O}(\mathsf{P})$-*filtered set*. The image $\mathsf{N} \subset \mathsf{Set}(X)$ of flt is a filtering on X. Common notation for an $\mathsf{O}(\mathsf{P})$-filtering is: $\alpha \mapsto F_\alpha X$. Given an $\mathsf{O}(\mathsf{P})$-filtering $\mathrm{flt}\colon \mathsf{O}(\mathsf{P}) \to \mathsf{Set}(X)$, with image N, then Birkhoff duality as described above yields the order-embedding $\iota\colon \mathsf{T}(\mathsf{N}) \hookrightarrow \mathsf{P}$ of the induced ordered tessellation:

$$T \overset{\iota}{\mapsto} \min\Big\{p \in \mathsf{P} \mid T \subset F_{\downarrow p} X\Big\}. \tag{C.1}$$

The latter defines a discretization map (not necessarily surjective, nor continuous) $\mathrm{grd}\colon X \to (\mathsf{P}, \leqslant)$ via:

$$\mathrm{grd}(x) := p, \quad x \in \iota^{-1}(p).$$

The discretization map grd is called a P-*grading* on X and the subsets $G_p X = \mathrm{grd}^{-1} p$ yield a decomposition

$$X = \bigcup_{p \in \mathsf{P}} G_p X, \tag{C.2}$$

which we refer to as a P-*graded decomposition* of X. The set $G_p X$ is also called the subsets of *homogeneous elements of degree* p. The non-empty sets $G_p X$ form a ordered tessellation of X. In this construction the P-grading grd is induced by the filtering flt. In general any discretization map $\mathrm{grd}\colon X \to \mathsf{P}$ yields a P-graded decomposition of X. Given a P-grading $\mathrm{grd}\colon X \to \mathsf{P}$, Birkhoff duality implies a lattice homomorphism $\mathrm{grd}^{-1}\colon \mathsf{O}(\mathsf{P}) \twoheadrightarrow \mathsf{O}(\mathsf{T})$. The formula

$$\mathrm{flt}\colon \mathsf{O}(\mathsf{P}) \to \mathsf{N}(\mathsf{T}), \quad \alpha \mapsto F_\alpha X := \bigcup\big\{T \mid T \in \mathrm{grd}^{-1}(\alpha)\big\}, \tag{C.3}$$

yields an $\mathsf{O}(\mathsf{P})$-filtering on X, which establishes teh duality between P-gradings and $\mathsf{O}(\mathsf{P})$-filterings of X.

C.2. P-graded and $\mathsf{O}(\mathsf{P})$-filtered modules

In the spirit of gradings and filterings of a set X we can do the same for R-modules. Let C be an R-module, or module for short, over a ring R. The submodule of C are denoted by $\mathsf{Sub}\, C$ with binary operations $\cap$ and $+$ (span). An $\mathsf{O}(\mathsf{P})$-*filtering* on C is a lattice homomorphism

$$\mathrm{flt}\colon \mathsf{O}(\mathsf{P}) \to \mathsf{Sub}\, C,$$

and the pair (C, flt) is called an $\mathsf{O}(\mathsf{P})$-*filtered* module. Common notation for an $\mathsf{O}(\mathsf{P})$-filtering is $\alpha \mapsto F_\alpha C$. For an $\mathsf{O}(\mathsf{P})$-filtering $\mathsf{O}(\mathsf{P}) \twoheadrightarrow \mathsf{Sub}\, C$ define the external direct sum

$$\mathrm{Gr}\, C = \bigoplus_{\alpha \in \mathsf{J}(\mathsf{O}(\mathsf{P}))} \frac{F_\alpha C}{F_{\alpha^\blacktriangleleft} C} \cong \bigoplus_{p \in \mathsf{P}} \frac{F_{\downarrow p} C}{F_{\downarrow p^\blacktriangleleft} C}, \tag{C.4}$$

where we use the fact that $F_\alpha C / F_\beta C \cong F_{\alpha'} C / F_{\beta'} C$ for all $\alpha \smallsetminus \beta = \alpha' \smallsetminus \beta'$. The module $\mathrm{Gr}\, C$ is the *associated graded module*, cf. [36] and [58]. In general $\mathrm{Gr}\, C$ is not isomorphic to C. If the subquotients $G_p C$ are free, then C is a free module and

$\mathrm{Gr}\, C \cong C$. This always holds if C is $\mathbb{K}$-vector space. The decomposition in (C.4) is called a P-*graded decomposition* (of $\mathrm{Gr}\, C$). The subquotients $G_pC := F_\alpha C/F_{\alpha^\blacktriangleleft} C$ are called *factors* and since flt is not necessarily injective some factors $F_\alpha C/F_{\alpha^\blacktriangleleft} C$ may be trivial, i.e. the zero module. In general a decomposition

$$C = \bigoplus_{p\in\mathsf{P}} G_pC, \quad G_pC = \frac{F_{\downarrow p}C}{F_{\downarrow p^\blacktriangleleft}C}, \tag{C.5}$$

is called a P-graded decomposition of C. The element in G_pC are called homogeneous elements of degree p. As before G_pC may be trivial for some p. Factors are also well-defined for any convex set $\beta \smallsetminus \alpha$ and are denoted by $G_{\beta\smallsetminus\alpha}C := F_\alpha C/F_\beta C$.

A P-graded module $C = \bigoplus_{p\in\mathsf{P}} G_pC$ yields an $\mathsf{O}(\mathsf{P})$-filtered module in a canonical way:

$$\mathrm{flt}\colon \mathsf{O}(\mathsf{P}) \to \mathsf{Sub}\, C, \quad \alpha \mapsto F_\alpha V := \bigoplus_{p\in\alpha} G_pC, \tag{C.6}$$

which is denoted by (C, flt). If C is a P-graded module then $\mathrm{Gr}\, C$ is defined as before via the induces $\mathsf{O}(\mathsf{P})$-filtered module and

$$\mathrm{Gr}\, C \cong C,$$

which establishes one of the dualities for arbitrary modules.

C.3. Differential modules

The concept of P-grading can be applied to chain complexes and differential modules/vector spaces. A *differential module* is a pair (C, d) where C is an R-module and $\mathrm{d}\colon C \to C$ is an endomorphism satisfying $\mathrm{d}^2 = 0$. If C is a vector space, then we refer to (C, d) as a *differential vector space*. We refer to the elements in C as *chains*. The chains in C for which d vanishes are called *cycles* and are denoted by $Z(C, \mathrm{d}) \subset C$. Chains in the range of d are called *boundaries* and are denoted by $B(C, \mathrm{d}) \subset Z$. The *homology* of (C, d) is define as $H(C, \mathrm{d}) := Z(C)/B(C)$. A homomorphism $h\colon C \to C'$ of R-modules is a *D-homomorphism* if $\mathrm{d}'h = h\mathrm{d}$.

DEFINITION C.3. A P-*graded differential module*[2] (C, d) is given by an P-graded module $C = \bigoplus_{p\in\alpha} G_pC$ such that

$$\mathrm{d}F_\alpha C \subset F_\alpha C, \quad \forall \alpha \in \mathsf{O}(\mathsf{P}), \tag{C.7}$$

which is equivalent to saying that d is $\mathsf{O}(\mathsf{P})$-filtered. A P-graded differential module C is *strict* if $\mathrm{d}|_{G_pC} = 0$ for all $p \in \mathsf{P}$. More generally an $\mathsf{O}(\mathsf{P})$-*filtered differential module* (C, d) is given by a $\mathsf{O}(\mathsf{P})$-filtered module C equipped with an $\mathsf{O}(\mathsf{P})$-filtered differential, i.e. d satisfies (C.7).

[2] A chain complex (C, d) is an $\mathbb{N}$-graded, or $\mathbb{Z}$-graded differential module with the differential d a degree -1 map, i.e. $\mathrm{d}F_{\downarrow p}C \subset F_{\downarrow(p-1)}C$. In the literature P-graded differential modules are also simply called *differential graded modules*.

The differential d on a P-graded module may be regarded as *upper-triangular* with entries $\mathrm{d}(p,q)\colon G_qC \to G_pC$ due (C.7). The latter implies

$$\mathrm{d}(p,q) \neq 0, \quad \Longrightarrow \quad p \leqslant q.$$

As before a P-graded differential module also yields an $\mathsf{O}(\mathsf{P})$-filtering on (C,d) making the latter an $\mathsf{O}(\mathsf{P})$-filtered differential module. The converse is not true in general. For example is we use field coefficients then an $\mathsf{O}(\mathsf{P})$-filtered differential module is isomorphic to a P-graded differential module as described in the previous section.

An $\mathsf{O}(\mathsf{P})$*-filtered D-homomorphism* between P-graded differential modules is a D-homomorphism $h\colon (C,\mathrm{d}) \to (C',\mathrm{d}')$, such that $h\colon C \to C'$ is an $\mathsf{O}(\mathsf{P})$-filtered homomorphism, i.e. $h\big(F_\alpha C\big) \subset F_\alpha C$ for all $\alpha \in \mathsf{O}(\mathsf{P})$.

REMARK C.4. In our treatment of graded and filtered differential modules we assume that the differential is filtered as well as the homomorphisms. This allows more flexibility. For example for a (co)chain complex the differential is homogeneous of degree ± 1. This implies that the differential is also filtered. The converse is not true.

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